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Correlations for Parameter-Dependent Random Matrices

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Correlations for parameter-dependent Gaussian random matrices, intermediate between symmetric and Hermitian and antisymmetric Hermitian and Hermitian, are calculated. The (dynamical) density–density correlation between eigenvalues at different values of the parameter is calculated for the symmetric to Hermitian transition and the scaled $N \to \infty$ limit is computed. For the antisymmetric Hermitian to Hermitian transition the equal-parameter n-point distribution function is calculated and the scaled limit computed. A circular version of the antisymmetric Hermitian to Hermitian transition is formulated. In the thermodynamic limit the equal-parameter distribution function is shown to coincide with the scaled-limit expression of this distribution for the Gaussian antisymmetric Hermitian to Hermitian transition. Furthermore, the thermodynamic limit of the corresponding density–density correlation is computed. The results for the correlations are illustrated by comparison with empirical correlations calculated from numerical data obtained from computer-generated Gaussian random matrices.

KEY WORDS: Random matrices; correlations; skew-orthogonal polynomials.

1. INTRODUCTION

Mehta and Pandey[(1)] introduced parameter-dependent Gaussian random matrices intermediate between symmetric and Hermitian, and anti-Hermitian and Hermitian, and calculated the corresponding equal-parameter n-point distribution function. In the former case the distributions were evaluated in the thermodynamic limit and subsequently shown (see e.g. ref. 2) to have

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observable consequences in models of quantum chaos with symmetry intermediate between orthogonal and unitary. In both cases, for finite N the one and two-point distribution can be empirically calculated from computer generated Gaussian random matrices (examples will be given below) and the theoretical predictions realized up to statistical accuracy.

In addition to the observable consequences of the results of Mehta and Pandey, the actual calculations have a number of noteworthy features. First, the problem of determining the joint distribution of the eigenvalues of parameter dependent random matrices is equivalent to solving a Fokker–Planck equation introduced by Dyson.(3) This Fokker–Planck equation can be interpreted as describing the Brownian motion evolution of a classical gas of particles repelling each other by a logarithmic potential, and confined to the neighborhood of the origin by a harmonic potential. Second, the expressions for the equal-parameter n-point distributions extend Mehta's quaternion determinant expressions(4) for the n-point distributions of Gaussian symmetric and Hermitian random matrices.

Further aspects of parameter-dependent Gaussian random matrices intermediate between symmetric and Hermitian, and anti-Hermitian and Hermitian are the topic of this paper. In addition to the equal-parameter distributions, a quantity of primary interest in the theory and applications of parameter-dependent random matrices is the (dynamical) density–density correlation ${}_d\rho_2^T((x_a, \tau_a), (x_b, \tau_b))$. For the circular ensemble COE to CUE transition this has been recently calculated in the thermodynamic limit to be given by [ref. 5; only the case $\tau_a = 0$ was explicitly considered, but a straightforward extension of the method gives the result for general τ_a]

$$\begin{aligned}
&{}_d\rho_{(2)}^{T(C)}((x_a, \tau_a), (x_b, \tau_b)) \\
&\quad = \rho^2 \Bigg(\int_1^\infty du_1\, e^{-(\tau_b - \tau_a)(\pi\rho u_1)^2/2} \cos \pi u_1 \rho(x_b - x_a) \\
&\qquad \times \int_0^1 du_2\, e^{(\tau_b - \tau_a)(\pi\rho u_2)^2/2} \cos \pi u_2 \rho(x_b - x_a) \\
&\qquad + \int_1^\infty du_1\, \frac{e^{-(\tau_a + \tau_b)(\pi\rho u_1)^2/2}}{u_1} \sin \pi u_1 \rho(x_b - x_a) \\
&\qquad \times \int_0^1 du_2\, u_2 e^{(\tau_a + \tau_b)(\pi\rho u_2)^2/2} \sin \pi u_2 \rho(x_b - x_a) \Bigg)
\end{aligned} \tag{1.1}$$

(the superscript (C) on the l.h.s. denotes the use of the circular ensemble, τ denotes the parameter and x denotes the eigenvalue). In Section 2, after a brief revision, ${}_d\rho^T_{(2)}$ will be calculated for the transition between symmetric and Hermitian random matrices. In the large-N limit, after appropriate scaling, the result (1.1) is reclaimed. Our method of calculation makes use of skew-orthogonal polynomials and functional differentiation, and we present formulas which are applicable to a wider class of Brownian motion problems.

In Section 3 the transition between Gaussian anti-symmetric Hermitian and Gaussian Hermitian (i.e. GUE) random matrices is considered. In particular, it is shown how the results of Mehta and Pandey for the equal-parameter distribution can be obtained in a systematic way by using the skew-orthogonal polynomials of the previous section and the general formulas of Frahm and Pichard[6] which express the equal-parameter distributions as quaternion determinants. Furthermore, the scaled $N \to \infty$ limit of the distributions is computed (this was not given by Mehta and Pandey).

Sections 4 and 5 address a circular ensemble version of the anti-symmetric Hermitian to Hermitian transition. In Section 4, using the general formalism of the previous section, the equal parameter distributions are expressed as quaternion determinants, and the thermodynamic limit is taken. Agreement with the expression obtained for the Gaussian case is found. In Section 5 the dynamical density–density correlation ${}_d\rho^{T(C)}_{(2)}((x_a, 0), (x_b, \tau_b))$ is computed in the thermodynamic limit.

2. DENSITY–DENSITY CORRELATION FOR THE GOE TO GUE TRANSITION

2.1. Revision

Before calculating the density–density correlation, we will briefly revise the definition of parameter-dependent Gaussian Hermitian random matrices, the relationship between the corresponding eigenvalue distribution and the Fokker–Planck equation, and the Green function solution of the Fokker–Planck equation.

Definition 2.1. A parameter-dependent Gaussian Hermitian random matrix H has all diagonal elements u_{jj} ($j = 1,..., N$) and upper diagonal elements $u_{jk} + iv_{jk}$ independently distributed with probability density functions

$$\frac{1}{\sqrt{\pi\,|1-e^{-2\tau}|}}\exp[-(u_{jj}-e^{-\tau}u^{(0)}_{jj})^2/|1-e^{-2\tau}|]$$

and

$$\frac{2}{\pi\,|1-e^{-2\tau}|}\exp[-2((u_{jk}-e^{-\tau}u_{jk}^{(0)})^2-(v_{jk}-e^{-\tau}v_{jk}^{(0)})^2)/|1-e^{-2\tau}|]$$

respectively.

We note from Definition 2.1 that the joint distribution of the elements of **H** is proportional to

$$\exp[-\mathrm{Tr}(\mathbf{H}-e^{-\tau}\mathbf{H}^{(0)})^2/|1-e^{-2\tau}|], \tag{2.1}$$

where $\mathbf{H}^{(0)}$ is the $N\times N$ Hermitian matrix with diagonal elements $u_{jj}^{(0)}$ and off diagonal elements $u_{jk}^{(0)}+iv_{jk}^{(0)}$, and Tr denotes the trace.

The joint distribution of the elements of **H** for a specific value of the parameter is obtained by averaging over the elements of $\mathbf{H}^{(0)}$. For example, if $\mathbf{H}^{(0)}$ is a random real symmetric matrix from the GOE, the distribution $P(u_{jj};\tau)$ of the diagonal elements u_{jj} is given by

$$\begin{aligned}P(u_{jj};\tau)&=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\pi\,|1-e^{-2\tau}|}}\int_{-\infty}^{\infty}du\,e^{-u^2/2}e^{-(u_{jj}-e^{-\tau}u)^2/|1-e^{-2\tau}|}\\&=\frac{1}{\sqrt{\pi\,|1+e^{-2\tau}|}}\exp-u_{jj}^2/|1+e^{-2\tau}|\end{aligned}\tag{2.2a}$$

while the distribution $P(u_{jk}+iv_{jk};\tau)$ of the off diagonal elements $u_{jk}+iv_{jk}$ is given by

$$\begin{aligned}P(u_{jk}+iv_{jk};\tau)&=\frac{1}{\sqrt{\pi}}\sqrt{\frac{2}{\pi\,|1-e^{-2\tau}|}}\,e^{-2v_{jk}^2/|1-e^{-2\tau}|}\\&\quad\times\int_{-\infty}^{\infty}du\,e^{-u^2}e^{-2(u_{jj}-e^{-\tau}u)^2/|1-e^{-2\tau}|}\\&=\frac{2}{\pi\sqrt{|1-e^{-4\tau}|}}\exp-2v_{jk}^2/|1-e^{-2\tau}|-2u_{jk}^2/|1+e^{-2\tau}|\end{aligned}\tag{2.2b}$$

Proposition 2.1. Let the eigenvalues of $\mathbf{H}^{(0)}$ have distribution $f(x_1^{(0)},\dots,x_N^{(0)})$ and denote by $F(x_1,\dots,x_N;\tau)$ the probability density function (p.d.f.) of the eigenvalues of **H** after averaging over f and the variables associated with the eigenvectors of **H**. Then F satisfies the Fokker–Planck equation

$$\frac{\partial F}{\partial \tau} = \mathscr{L} F \qquad \text{where} \quad \mathscr{L} = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial x_j} + \frac{1}{2}\frac{\partial}{\partial x_j}\right)$$

with

$$W = -\sum_{1 \leqslant j < k \leqslant N} \log |x_j - x_k| + \tfrac{1}{2} \sum_{j=1}^{N} x_j^2$$

subject to the initial condition

$$F(x_1, \ldots, x_N; 0) = f(x_1, \ldots, x_N).$$

Proof. This result follows by changing variables in the Fokker–Planck equation satisfied by (2.1) (see e.g. refs. 7, Appendix 5 and ref. 8).

Proposition 2.2. The Green function solution $G := G(x_1^{(0)}, \ldots, x_N^{(0)} \mid x_1, \ldots, x_N;\, \tau)$ of the Fokker–Planck equation in Proposition 2.1 is given by

$$G = e^{N^2 \tau} \prod_{1 \leqslant j < k \leqslant N} \frac{x_k - x_j}{x_k^{(0)} - x_j^{(0)}} \prod_{l=1}^{N} e^{-(x_l^2 - (x_l^{(0)})^2)/2} \det[\, g(x_j^{(0)}, x_k; \tau)\,]_{j,k=1,\ldots,N} \tag{2.3a}$$

where it is assumed $x_1 \geqslant \cdots \geqslant x_N$ and similarly $x_1^{(0)} \geqslant \cdots \geqslant x_N^{(0)}$, and where

$$\begin{aligned} g(x, y; \tau) &= e^{-(x^2 + y^2)/2} \sum_{n=0}^{\infty} \frac{H_n(x)\, H_n(y)}{\sqrt{\pi}\, 2^n n!} e^{-(n+1/2)\tau} \\ &= e^{-y^2/2 + x^2/2} \frac{e^{-\tau/2}}{\sqrt{\pi(1 - e^{-2\tau})}} \exp[-(x - y e^{-\tau})^2/(1 - e^{-2\tau})] \end{aligned} \tag{2.3b}$$

with $H_n(x)$ denoting the Hermite polynomial of degree n.

Proof. This follows since

$$-e^{W} \mathscr{L} e^{-W} = H - N^2$$

where

$$W = -\sum_{1 \leqslant j < k \leqslant N} \log |x_k - x_j| + \frac{1}{2} \sum_{j=1}^{N} x_j^2 \quad \text{and} \quad H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{N} x_j^2$$

(see e.g. ref. 9).

2.2. General Formulas

The (dynamical) density–density correlation is defined in terms of the initial distribution f and the Green function G by

$$
\begin{aligned}
&{}_d\rho^T_{(2)}((x_a, \tau_a), (x_b, \tau_b)) \\
&\quad = \int_I dx_1^{(0)} \cdots \int_I x_N^{(0)} f(x_1^{(0)},..., x_N^{(0)}) \\
&\qquad \times \int_I dx_1^{(1)} \cdots \int_I dx_N^{(1)} G(x_1^{(0)},..., x_N^{(0)} \mid x_1^{(1)},..., x_N^{(1)}; \tau_a) \sum_{l=1}^{N} \delta(x_l^{(1)} - x_a) \\
&\qquad \times \int_I dx_1^{(2)} \cdots \int_I dx_N^{(2)} G(x_1^{(1)},..., x_N^{(1)} \mid x_1^{(2)},..., x_N^{(2)}; \tau_b - \tau_a) \sum_{l=1}^{N} \delta(x_l^{(2)} - x_b) \\
&\qquad - {}_d\rho_{(1)}(x_a; \tau_a)\, {}_d\rho_{(1)}(x_b; \tau_b)
\end{aligned} \tag{2.4a}
$$

where

$$
\begin{aligned}
{}_d\rho_{(1)}(x; \tau) := &\int_I dx_1^{(0)} \cdots \int_I dx_N^{(0)} f(x_1^{(0)},..., x_N^{(0)}) \\
&\times \int_I dx'_1 \cdots \int_I dx'_N\, G(x_1^{(0)},..., x_N^{(0)} \mid x'_1,..., x'_N; \tau) \sum_{l=1}^{N} \delta(x'_l - x)
\end{aligned} \tag{2.4b}
$$

and $I = (-\infty, \infty)$.

This can be computed by introducing a generalized partition function

$$
\begin{aligned}
&Z[a, b] \\
&\quad := \int_I dx_1^{(0)} \cdots \int_I dx_N^{(0)} f(x_1^{(0)},..., x_N^{(0)}) \\
&\qquad \times \frac{1}{N!} \int_I dx_1^{(1)} a(x_1^{(1)}) \cdots \int_I dx_N^{(1)} a(x_N^{(1)})\, G(x_1^{(0)},..., x_N^{(0)} \mid x_1^{(1)},..., x_N^{(1)}; \tau_a) \\
&\qquad \times \frac{1}{N!} \int_I dx_1^{(2)} b(x_1^{(2)}) \cdots \int_I dx_N^{(2)} b(x_N^{(2)})\, G(x_1^{(1)},..., x_N^{(1)} \mid x_1^{(2)},..., x_N^{(2)}; \tau_b - \tau_a)
\end{aligned} \tag{2.5}
$$

and using functional differentiation. Thus from the defining formula

$$
\frac{\delta}{\delta a(x)} \int_{-\infty}^{\infty} a(y)\, f(y)\, dy = f(x) \tag{2.6}
$$

and the normalization property of the Green function

$$\frac{1}{N!}\int_I dx'_1 \cdots \int_I dx'_N\, G(x_1,..., x_N | x'_1,..., x'_N; \tau) = 1 \tag{2.7}$$

we see that

$${}_d\rho^T_{(2)}((x_a, \tau_a), (x_b, \tau_b)) = \frac{\delta^2}{\delta a(x_a)\, \delta b(x_b)} \log Z[a, b]|_{a=b=1} \tag{2.8}$$

Our task is therefore to compute (2.5) and (2.8) with G given by (2.3) and

$$f(x_1^{(0)},..., x_N^{(0)}) = \frac{1}{C_N} \prod_{l=1}^{N} e^{-x_l^2/2} \prod_{1 \leqslant j<k \leqslant N} |x_k - x_j| \tag{2.9}$$

which is the eigenvalue p.d.f. of the GOE (see e.g. ref 7).

To put this task in a more general setting, let $w_2(x)$ be an arbitrary weight function with support in I, let $\{p_j(x)\}_{j=0,1,...}$ denote the polynomials orthonormal with respect to the weight function $w_2(x)$. We see that the Green function (2.3) can then be written

$$G(x_1^{(0)},..., x_N^{(0)} | x_1,..., x_N; \tau) = e^{E_0\tau} \prod_{j=1}^{N} \frac{w_1(x_j)}{w_1(x_j^{(0)})} \prod_{1 \leqslant j<k \leqslant N} \frac{x_k - x_j}{x_k^{(0)} - x_j^{(0)}} \det[g(x_j^{(0)}, x_k; \tau)]_{j,k=1,...,N} \tag{2.10a}$$

where

$$g(x^{(0)}, x; \tau) = w_1(x^{(0)})\, w_1(x) \sum_{j=0}^{\infty} p_j(x^{(0)})\, p_j(x)\, e^{-\nu_j \tau} \tag{2.10b}$$

with

$$w_1(x) = (w_2(x))^{1/2} = e^{-x^2/2}, \quad p_j(x) = \left(\frac{1}{\sqrt{\pi}\, 2^j j!}\right)^{1/2} H_j(x), \quad \nu_j = 1 + 1/2 \tag{2.11}$$

and the initial p.d.f. can be written,

$$f(x_1,..., x_N) = \frac{1}{C'_{1N}} \prod_{j=1}^{N} w_1(x_j) \prod_{1 \leqslant j<k \leqslant N} |x_k - x_j| \tag{2.12}$$

To calculate the corresponding density–density correlation define

$$H_1(x_1, x_2; \tau) := \tfrac{1}{2}\int_I dy_1 \int_I dy_2 \, \mathrm{sgn}(y_2 - y_1)\, g(y_1, x_1; \tau)\, g(y_2, x_2; \tau) \tag{2.13}$$

and introduce the skew-symmetric inner product

$$\langle f \mid g \rangle_s^{(\tau)} := \int_I dy_1\, w_1(y_1) \int_I dy_2\, w_1(y_2)\, H_1(y_1, y_2; \tau)\, f(y_1)\, g(y_2) \tag{2.14}$$

The polynomials $\{U_n(x; \tau)\}_{n=0,1,\dots}$ are said to be skew orthogonal with respect to the inner product (2.14) if

$$\langle U_{2m}(\cdot; \tau) \mid U_{2n+1}(\cdot; \tau) \rangle_s^{(\tau)} \, (= -\langle U_{2n+1}(\cdot; \tau) \mid U_{2m}(\cdot; \tau) \rangle_s^{(\tau)}) = u_m \, \delta_{m,n} \tag{2.15a}$$

and

$$\langle U_{2m}(\cdot; \tau) \mid U_{2n}(\cdot; \tau \rangle_s^{(\tau)} = \langle U_{2m+1}(\cdot; \tau) \mid U_{2n+1}(\cdot; \tau) \rangle_s^{(\tau)} = 0 \tag{2.15b}$$

(the quantity u_m is referred to as the normalization and $\delta_{m,n}$ denotes the Kronecker delta). It will further be assumed that the polynomials are monic i.e. the coefficient of the highest power is unity. The polynomials for general τ can be constructed from the polynomials with $\tau = 0$.[6] Thus, given

$$U_n(x; 0) = \sum_{k=0}^{n} a_k\, p_k(x) \tag{2.16a}$$

from the orthonormality property of $\{p_k(x)\}_{k=0,1,\dots}$ it is straightforward to check that by defining

$$U_n(x; \tau) := \sum_{k=0}^{n} a_k\, p_k(x)\, e^{v_k \tau} \tag{2.16b}$$

we have

$$\langle U_m(\cdot; \tau) \mid U_n(\cdot; \tau) \rangle_s^{(\tau)} = \langle U_m(\cdot; 0) \mid U_n(\cdot; 0) \rangle_s^{(0)} \tag{2.16c}$$

and thus the polynomials (2.15b) are skew orthogonal with respect to the inner product (2.14).

In the particular case of interest, when the initial state is (2.9) which is the eigenvalue p.d.f for the GOE, we know (see e.g. ref. 10) that at $\tau = 0$

$$U_{2m}(x; 0) = 2^{-2m} H_{2m}(x),$$
$$U_{2m+1}(x; 0) = -e^{x^2/2} \frac{d}{dx}(e^{-x^2/2} U_{2m}(x; 0)) \tag{2.17a}$$
$$= 2^{-2m-1} H_{2m+1}(x) - m 2^{-2m+1} H_{2m-1}(x)$$

with

$$u_n = 2^{-2n} \sqrt{\pi}\,(2n)! \tag{2.17b}$$

Consequently, from (2.15b). and (2.11) we have

$$U_{2m}(x; \tau) = 2^{-2m} e^{(2m+1/2)\tau} H_{2m}(x) \tag{2.17c}$$

and

$$U_{2m+1}(x; \tau) = 2^{-2m-1} e^{(2m+3/2)\tau} H_{2m+1}(x) - m 2^{-2m+1} e^{(2m-1/2)\tau} H_{2m-1}(x) \tag{2.17d}$$

and the normalization for general τ is still given by (2.17b).

Using the skew orthogonal polynomials, the formulas (2.5) and (2.8) can be computed.

Proposition 2.3. For N even we have

$$Z[a, b] = \frac{2^{N/2}}{C'_{1N}} \det[H_{jk}]^{1/2}_{j, k=1,\ldots, N}$$

where

$$H_{jk} = \tfrac{1}{2} \int_I dx \int_I dx'\, A_j(x)\, A_k(x')\, \mathrm{sgn}(x - x')$$

with

$$A_j(x) = \int_I dy\, a(y)\, g(x, y; \tau_a)\, B_j(y)$$

$$B_j(y) = \int_I dv\, w(v)\, b(v)\, g(v, y; \tau_b - \tau_a)\, U_{j-1}(v; \tau_b)$$

This gives

$$
\begin{aligned}
&{}_d\rho^T_{(2)}((x_a, \tau_a), x_b, \tau_b)) \\
&\quad = \sum_{j=0}^{N/2-1} \frac{K_{2j+1,2j+2}((x_a, \tau_a), (x_b, \tau_b))}{u_j} \\
&\quad - \sum_{j,j'=0}^{N/2-1} \left(\frac{I_{2j+1,2j'+2}(x_a, \tau_a)\, I_{2j'+1,2j+2}(x_b, \tau_b)}{u_j u_{j'}} \right. \\
&\quad - \frac{I_{2j+2,2j'+2}(x_a, \tau_a)\, J_{2j'+1,2j+1}(x_b, \tau_b)}{2u_j u_{j'}} \\
&\quad \left. - \frac{I_{2j+1,2j'+1}(x_a, \tau_a)\, J_{2j'+2,2j+2}(x_b, \tau_b)}{2u_j u_{j'}} \right)
\end{aligned}
$$

where

$$
\begin{aligned}
I_{j,k}(x_a, \tau_a) &= \frac{\delta}{\delta a(x_a)} H_{jk}|_{a=b=1} = w_1(x_a)\, U_{j-1}(x_a; \tau_a) \\
&\quad \times \int_I dx\, g(x, x_a; \tau_a)\, \Phi_{k-1}(x) - (j \leftrightarrow k) \\
J_{j,k}(x_b, \tau_b) &= \frac{\delta}{\delta b(x_b)} H_{jk}|_{a=b=1} = I_{j,k}(x_b, \tau_b)
\end{aligned}
$$

and

$$
\begin{aligned}
&K_{j,k}((x_a, \tau_a), (x_b, \tau_b)) \\
&\quad = \frac{\delta^2}{\delta a(x_a)\, \delta b(x_b)} H_{jk}|_{a=b=1} \\
&\quad = \frac{1}{2} w_1(x_a)\, w_1(x_b)\, U_{j-1}(x_a; \tau_a)\, U_{k-1}(x_b; \tau_b) \int_I dx \\
&\quad \times \int_I dx' g(x, x_a; \tau_a)\, g(x', x_b; \tau_b)\, \mathrm{sgn}(x - x') - (j \leftrightarrow k) \\
&\quad + w_1(x_a)\, w_1(x_b)\, g(x_a, x_b; \tau_b - \tau_a)\, U_{j-1}(x_b; \tau_b) \\
&\quad \times \int_I dx g(x, x_a; \tau)\, \Phi_{k-1}(x) - (j \leftrightarrow k)
\end{aligned}
$$

with

$$\Phi_k(x) := \frac{1}{2}\int_I dy\, w_1(y)\,\mathrm{sgn}(x-y)\, U_k(y;0)$$

Proof. (Sketch) The formula for $Z[a, b]$ is obtained from the general formula (2.5) by using the method of integration over alternate variables,[7] while the formula for ${}_a\rho^T_{(2)}$ follows from the formula for $Z[a, b]$ by applying (2.8) and noting that

$$B_j(y)|_{b=1} = w_1(y)\, U_{j-1}(y;\tau_a)$$

By the skew orthogonality property of $\{U_j(x;\tau)\}_{j=0,1,\dots}$ this implies that for $a=b=1$

$$H_{2m+1\ 2n+1} = -H_{2n+1\ 2m+1} = u_n\,\delta_{m,n}, \qquad H_{2m\ 2n} = H_{2m+1\ 2n+1} = 0$$

which makes the computation of the determinant straightforward.

Consider the GOE initial state (2.9). The skew orthogonal polynomials at $\tau=0$ are given by (2.17a), so from the definition of $\Phi_n(x)$ in Proposition 2.3 we see that

$$\Phi_{2k}(x) = 2^{-2k}\int_0^x H_{2k}(y)\, e^{-y^2/2}\,dy \quad \Phi_{2k+1}(x) = -2^{-2k}e^{-x^2/2}H_{2k}(x) \qquad (2.18)$$

All the quantities in Proposition 2.3 are therefore known explicitly. Our immediate task is therefore to compute the scaled $N\to\infty$ limit, where the scaling required is[5]

$$x \mapsto \pi\rho x/\sqrt{2N}, \qquad \tau \mapsto (\pi\rho)^2\,\tau/2N \qquad (2.19)$$

In the limit (2.19) the Fokker–Planck equation in Proposition 2.1 approaches the Fokker–Planck equation describing transitions to the circular ensemble with unitary symmetry (see Section 4), so it is expected that the correlations will agree with those of the circular ensemble description in the thermodynamic limit. We remark that Proposition 2.3 requires modification for N odd. However, since we are specifically interested in the $N\to\infty$ limit and there is no reason to expect that the parity of N will affect the limit, it suffices to consider the N even case. (This is not true of the anti-Hermitian to Hermitian transition considered in Section 3, as the functional form of the eigenvalue p.d.f. of a random anti-symmetric matrix depends on the parity of N.)

2.3. The Scaled $N \to \infty$ Limit

To compute the scaled $N \to \infty$ limit of the sums in Proposition 2.3 we first compute the asymptotic form of the summands. This approach is justified by Theorem 3.1 of ref. 11. To compute the asymptotic form of the summands we use the large-n asymptotic formula

$$\frac{\Gamma(n/2+1)}{\Gamma(n+1)} e^{-x^2/2} H_n(x) \sim \cos(\sqrt{2n+1}\, x - n\pi/2)$$

in the formulas (2.3), (2.17c) and (2.18) for $g(x, y; \tau)$, $U_n(x; \tau)$ and $\Phi_k(x)$ respectively. This gives

$$w_1(x) \frac{U_{2n+1}(x; \tau)}{u_n^{1/2}} \sim \frac{(-1)^n}{\pi^{1/4}} \left(\frac{N}{2}\right)^{1/4} \left(\frac{t}{\pi}\right)^{1/4} e^{t\tau(\pi\rho)^2/2} 2 \sin(\sqrt{t}\, \pi\rho x) \tag{2.20a}$$

$$w_1(x) \frac{U_{2n}(x; \tau)}{u_n^{1/2}} \sim \frac{(-1)^n}{\pi^{1/4}} \left(\frac{2}{N}\right)^{1/4} \frac{e^{t\tau(\pi\rho)^2/2}}{(\pi t)^{1/4}} \cos(\sqrt{t}\, \pi\rho x) \tag{2.20b}$$

$$\frac{\Phi_{2n+1}(x)}{u_n^{1/2}} \sim -\frac{(-1)^n}{\pi^{1/4}} \left(\frac{2}{N}\right)^{1/4} \frac{\cos(\sqrt{t}\, \pi\rho x)}{(\pi t)^{1/4}} \tag{2.20c}$$

$$\frac{\Phi_{2n}(x)}{u_n^{1/2}} \sim \frac{(-1)^n}{\pi^{1/4}} \left(\frac{2}{N}\right)^{1/4} \frac{\pi\rho}{\sqrt{2N}} \frac{1}{(\pi t)^{1/4}} \frac{\sin(\sqrt{t}\, \pi\rho x)}{\sqrt{t}\, \pi\rho} \tag{2.20d}$$

$$g(x, y; \tau) \sim \frac{1}{\pi^{1/2}} \left(\frac{N}{(\pi\rho)^2 \tau}\right)^{1/2} \exp[-(x-y)^2/2\tau]$$

$$= \frac{(2N)^{1/2}}{\pi\rho} \rho \int_{-\infty}^{\infty} du\, e^{-2(\pi\rho)^2 u^2 \tau} \cos 2\pi\rho(x-y)\, u \tag{2.20e}$$

where $t := 2n/N$.

Consequently, from the formulas in Proposition 2.3

$$\frac{J_{2j'+2,2j+1}(x; \tau)}{r_{j'}^{1/2} r_j^{1/2}}$$

$$\sim \frac{(-1)^{j+j'}}{\pi} \sqrt{\frac{2}{N}} \left(t'^{1/4} e^{t'\tau(\pi\rho)^2/2} \sin(\sqrt{t'}\, \pi\rho x) \frac{e^{-t\tau(\pi\rho)^2/2}}{t^{3/4}} \sin(\sqrt{t}\, \pi\rho x) \right.$$

$$\left. + \frac{e^{t\tau(\pi\rho)^2/2} e^{-t'\tau(\pi\rho)^2/2}}{t^{1/4} t'^{1/4}} \cos(\sqrt{t'}\, \pi\rho x) \cos(\sqrt{t}\, \pi\rho x) \right) \tag{2.21a}$$

$$\frac{J_{2j+1,2j'+1}(x;\tau)}{r_{j'}^{1/2}r_j^{1/2}}$$
$$\sim\frac{(-1)^{j+j'}}{\pi}\frac{2}{N}\left(\frac{e^{t\tau(\pi\rho)^2/2}e^{-t'\tau(\pi\rho)^2/2}}{t^{1/4}t'^{3/4}}\sin(\sqrt{t'}\,\pi\rho x)\right.$$
$$\left.\times\cos(\sqrt{t}\,\pi\rho x)-(t\leftrightarrow t')\right) \tag{2.21b}$$

$$\frac{J_{2j'+2,2j'+2}(x;\tau)}{r_{j'}^{1/2}r_j^{1/2}}$$
$$\sim-\frac{(-1)^{j+j'}}{\pi}\left(\frac{e^{t'\tau(\pi\rho)^2/2}e^{-t\tau(\pi\rho)^2/2}}{(\pi t)^{1/4}}t'^{1/4}\sin(\sqrt{t'}\,\pi\rho x)\right.$$
$$\left.\times\cos(\sqrt{t}\,\pi\rho x)-(t\leftrightarrow t')\right) \tag{2.21c}$$

$$\frac{K_{2j+1,2j+2}((x_a,\tau_a),(x_b,\tau_b))}{u_j}$$
$$\sim\frac{1}{\pi^2}e^{t(\tau_a+\tau_b)(\pi\rho)^2/2}\sin\sqrt{t}\,\pi\rho(x_b-x_a)\int_{-\infty}^{\infty}du$$
$$\times\frac{e^{-2(\tau_a+\tau_b)(\pi\rho u)^2}}{u}\sin 2\pi u\rho(x_b-x_a)$$
$$+\frac{2}{\pi^2}e^{t(\tau_b-\tau_a)(\pi\rho)^2/2}\cos\pi\rho\sqrt{t}\,(x_b-x_a)$$
$$\times\int_{-\infty}^{\infty}du\,e^{-2(\tau_a-\tau_b)(\pi\rho u)^2}\cos 2\pi u\rho(x_b-x_a) \tag{2.21d}$$

where use has been made of the integration formula in (2.20e) and the further integration formulas

$$\frac{1}{(2\pi\tau)^{1/2}}\int_{-\infty}^{\infty}dx\,e^{-(x-x_b)^2/2\tau}\operatorname{sgn}(x-x')$$
$$=-\frac{1}{\pi}\int_{-\infty}^{\infty}du\frac{e^{-2\tau(\pi\rho u)^2}}{u}\sin 2\pi u\rho(x_b-x') \tag{2.22a}$$

and

$$\frac{1}{(2\pi\tau)^{1/2}}\int_{-\infty}^{\infty} dx\, e^{-(x-x_a)^2/2\tau}\sin 2\pi u\rho(x_b - x) = e^{-2\tau(\pi u\rho)^2}\sin 2\pi u\rho(_b - x_a) \tag{2.22b}$$

With the asymptotic formulas (2.21) we see that the sums in Proposition 2.3 are Riemann approximations to definite integrals. Straightforward simplification gives

$$\lim_{N\to\infty}\left(\frac{\pi\rho}{\sqrt{N}}\right)^2 {}_d\rho_{(2)}^{T(G)}((\pi\rho x_a/\sqrt{2N}, (\pi\rho)^2\tau_a/2N), (\pi\rho x_b/\sqrt{2N}, (\pi\rho)^2\tau_b/2N))$$
$$= {}_d\rho_{(2)}^{T(C)}((x_a, \tau_a), (x_b, \tau_b)) \tag{2.23}$$

where ${}_d\rho_{(2)}^{T(C)}$ is given by (1.1). Thus, as expected, the scaled $N\to\infty$ limit of ${}_d\rho_{(2)}^{T}$ for the GOE → GUE transition agrees with the thermodynamic limit of ${}_d\rho_{(2)}^{T}$ for the COE → CUE transition.

To illustrate and to provide a realization of (2.23), we have performed an empirical computation of ${}_d\rho_{(2)}^{T(G)}((\pi\rho x_a/\sqrt{2N}, 0), (\pi\rho_b/\sqrt{2N}, \tau_b))$ for the eigenvalues of 10,000 pairs of 13 × 13 computer generated matrices (the value $N = 13$ is large enough to probe the $N\to\infty$ behavior in the middle of the spectrum). The first member of each pair is a random matrix from the GOE, with elements specified according to (2.2) with $\tau = 0$. The second member is constructed from the first according to Definition 2.1 with a certain value of $\tau = \tau_b$. The eigenvalues of both matrices are calculated, multiplied by the scale $\pi/(2N-(\pi x)^2)^{1/2}$ (the reciprocal of the density at point x) so that the mean spacing is unity, and labelled -6 up to 6 sequentially along the real axis. The distance from the middle eigenvalue (label 0) of the first matrix to each of the eigenvalues of the second matrix with labels $-2,\ldots,2$ are recorded and used to compute the probability density functions $p(j; X)$ for the event that the distance X is in the interval $[X, X+dx]$ (dx is taken as 0.1).

To relate this to the density–density correlation suppose the eigenvalue with label 0 of the first matrix occurs in some interval dx of the origin. Then

$${}_d\rho_{(2)}^{T}((0,0), (X, \tau_b)) = -1 + \sum_{-(N-1)/2}^{(N-1)/2} P(j; X) \tag{2.24}$$

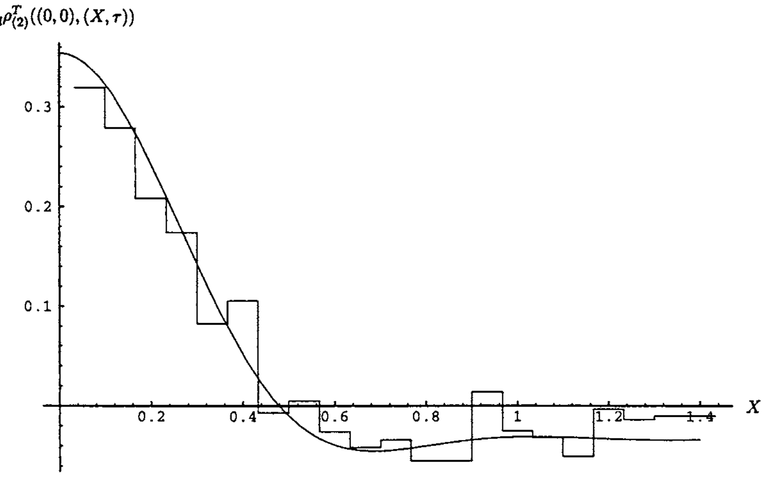

Fig. 1. Comparison between $_d\rho^T_{(2)}((0,0),(X,\tau))$, $\tau=0.05$, as calculated empirically from 10,000 computer generated parameter-dependent random matrices of dimension 13×13 for the GOE → GUE transition, and the theoretical prediction for the scaled limit of the same quantity.

where $P(j;X)$ denotes the probability density function for the event that the scaled eigenvalue with label j of the second matrix occurs within the interval $[X, X+dx]$. In (2.24) we make the approximation

$$P(-j;X)+P(j;X)\approx\tfrac{1}{2}(p(-j;X)+p(j;X)) \tag{2.25}$$

which is exact in the $N\to\infty$ limit when there is translation invariance, to compute an empirical approximation to $_d\rho^T_{(2)}$. The result of this computation for $\tau=0.05$ is given as a bar graph in Fig. 1 and compared with the theoretical curve as given by (1.1), with $\rho=1$, $\tau_a=0$ and $\tau_b=2N\tau/\pi^2$ (the latter equation follows from (2.19)).

3. THE GAUSSIAN ANTI-SYMMETRIC HERMITIAN TO GUE TRANSITION

3.1. Gaussian Anti-symmetric Hermitian Random Matrices

Gaussian anti-symmetric Hermitian random matrices are zero along the diagonal, and have zero real part for the off diagonal elements. The imaginary part of the off diagonal elements $v_{jk}^{(0)} (j<k)$ are distributed according to the Gaussian probability density

$$\sqrt{\frac{1}{\pi}} \exp[-(v_{jk}^{(0)})^2] \tag{3.1}$$

Averaging over this distribution in Definition 2.1 gives that the p.d.f. of the elements of **H** for a specific value of the parameter τ in the Gaussian anti-symmetric Hermitian to GUE transition is given by (c.f. (2.2))

$$\frac{1}{\sqrt{\pi \,|1-e^{-2\tau}|}} \exp[-u_{jj}^2/|1-e^{-2\tau}|] \tag{3.2a}$$

and

$$\frac{2}{\pi \sqrt{|1-e^{-4\tau}|}} \exp[-2u_{jk}^2/|1-e^{-2\tau}| - 2v_{jk}^2/(1+e^{-2\tau})] \tag{3.2b}$$

for the diagonal and off diagonal elements respectively. The eigenvalue distribution of Gaussian anti-symmetric Hermitian random matrices depends on the parity of N. For both N even and N odd the eigenvalues come in pairs $\pm x_1,..., \pm x_{[N/2]}$, but for N odd there is also an eigenvalue at $x=0$. The independent eigenvalues $x_1,..., x_{[N/2]}$ have p.d.f.

$$d(x_1,..., x_{[N/2]}) = \frac{1}{C'_N} \prod_{j=1}^{[N/2]} e^{-x_j^2} \chi_N(x_j) \prod_{1 \leqslant j<k \leqslant [N/2]} (x_k - x_j)^2 \tag{3.3}$$

where $\chi_N(x_j) := 1$ for N even and $\chi_N(x_j) = x_j^2$ for N odd. Thus, unlike the GOE $\to$ GUE transition, the correlations describing the Gaussian anti-symmetric Hermitian to GUE transition will depend on the parity of N, so the N even and N odd cases must be considered separately.

3.2. Equal Parameter Distribution Function

In general, for a given initial eigenvalue p.d.f. $f(x_1^{(0)},..., x_N^{(0)})$, the eigenvalue p.d.f. at parameter value τ, $p(x_1,..., x_N; \tau)$ say, is given in terms of the Green function by

$$p(x_1,..., x_N; \tau)$$
$$= \frac{1}{N!} \int_I dx_1^{(0)} \cdots \int_I dx_N^{(0)} f(x_1^{(0)},..., x_N^{(0)})\, G(x_1^{(0)},..., x_N^{(0)} \,|\, x_1,..., x_N; \tau) \tag{3.4}$$

The equal parameter distribution functions $\rho_n(x_1,..., x_n; \tau)$ are defined in terms of $p(x_1,..., x_N; \tau)$ by

$$\rho_{(n)}(x_1,..., x_n; \tau) := N(N-1) \cdots (N-n+1) \times \int_I dx_{n+1} \cdots \int_I dx_N \, p(x_1,..., x_N; \tau) \tag{3.5}$$

It is our task in this section to calculate $\rho_{(n)}$ when the initial conditions are given by

$$f(x_1,..., x_N) = d(x_1,..., x_{[N/2]}) \prod_{j=1}^{[N/2]} \delta(x_j - x_{N+1-j}) \, \tilde{\chi}_N(x_{(N+1)/2}) \tag{3.6}$$

where $\tilde{\chi}_N(x) = 1$ for N even and $\tilde{\chi}_N(x) = \delta(x)$ for N odd ($\delta(x)$ denotes the Dirac delta function). The first step is to write (3.4) as a Pfaffian.

Proposition 3.1. Let the initial eigenvalue p.d.f. be given by (3.6) and suppose G is given by (2.3). Then for N even the expression (3.4) can be written as

$$p(x_1,..., x_N; \tau) = \frac{1}{C'_{1N}} e^{E_0 \tau} \prod_{j=1}^{N} w_1(x_j) \prod_{1 \leq j < k \leq N} (x_k - x_j) \, \mathrm{Pf}[H_a(x_j, x_k; \tau)]_{j;k=1,...,N}$$

where

$$H_a(x_1, x_2; \tau) := \frac{1}{2} \int_0^\infty \frac{du}{u} (g(u, x; \tau) \, g(-u, y; \tau) - g(u, y; \tau) \, g(-u, x; \tau))$$

and $E_0 = N^2$. This formula remains valid for N odd, provided the Pfaffian is replaced by

$$2 \, \mathrm{Pf} \begin{pmatrix} [H_a(x_j, x_k)]_{j,k=1,...,N} & [F(x_j; \tau)]_{j=1,...,N} \\ -[F(x_k; \tau)]_{k=1,...,N} & 0 \end{pmatrix}$$

where

$$F(x; \tau) := \tfrac{1}{2} g(0, x; \tau)$$

Proof. In the limits $x_{N+1-j} \to x_j$ $(j=1,..., [N/2])$, $x_{(N+1)/2} \to 0$ (N odd) the Green function (2.3) multiplied by the distribution (3.3) reads

$$d(x_1^{(0)},..., x_{[N/2]}^{(0)})\, G(x_1^{(0)},..., x_N^{(0)} | x_1,..., x_N; \tau)$$
$$= \frac{e^{N^2\tau}}{\prod_{j=1}^{[N/2]} 2x_j} \prod_{j=1}^{N} w_1(x_j) \prod_{1 \leq j < k \leq N} (x_k - x_j) \det \mathbf{A}_N$$

where $w_1(x) = e^{-x^2/2}$ and $\mathbf{A}_N$ is given by

$$\begin{bmatrix} g(x_j^{(0)}, x_k; \tau) \\ g(-x_j^{(0)}, x_k; \tau) \end{bmatrix}_{\substack{j=1,..., [N/2] \\ k=1,..., N}} \quad \text{and} \quad \begin{bmatrix} g(x_j^{(0)}, x_k; \tau) \\ g(-x_j^{(0)}, x_k; \tau) \\ g(0, x_k; \tau) \end{bmatrix}_{\substack{j=1,..., [N/2] \\ k=1,..., N}}$$

for N even and N odd respectively. With this structure the problem of integrating over $x_1^{(0)},..., x_{[N/2]}^{(0)}$ is of a known type (ref. 7, Appendix A.18), and the Pfaffian structure results.

The next step is to introduce an appropriate skew symmetric inner product:

$$\langle f \mid g \rangle_s^{(\tau)} := \int_I dy_1\, w_1(y_1) \int_I dy_2\, w_1(y_2)\, H_a(y_1, y_2; \tau)\, f(y_1)\, g(y_2) \tag{3.7}$$

(c.f. (2.14)). Note that in the limit $\tau \to 0$, since $g(x, y, \tau) \to \delta(x-y)$ this reduces to

$$\langle f \mid g \rangle_s^{(0)} = \frac{1}{2} \int_0^\infty \frac{dx}{x} e^{-x^2} (f(x)\, g(-x) - f(-x)\, g(x)) \tag{3.8}$$

From the theory of Section 2.1 we know that the general τ-dependent skew orthogonal polynomials can be written down once the $\tau = 0$ skew orthogonal polynomials are known in terms of the Hermite polynomials.

Now, from the orthogonality of the Hermite polynomials, it is easy to check that

$$\begin{aligned} U_{2k}(x; 0) &= H_{2k}(x), \\ U_{2k+1}(x; 0) &= 2xH_{2k}(x) = H_{2k+1}(x) + 4kH_{2k-1}(x) \end{aligned} \tag{3.9}$$

with corresponding normalization

$$u_n = -2^{2n} \sqrt{\pi}\, (2n)! \tag{3.10}$$

are skew orthogonal with respect to the $\tau = 0$ inner product (3.8). Thus the polynomials

$$U_{2k}(x; \tau) = e^{(2k+1/2)\tau} H_{2k}(x) \qquad (3.11a)$$

$$\begin{aligned} U_{2k+1}(x; \tau) &= e^{(2k+3/2)\tau} H_{2k+1}(x) + 4k e^{(2k-1/2)\tau} H_{2k-1}(x) \\ &= -(1 - e^{-2\tau})\, e^{(2n+3/2)\tau} e^{x^2/(1-e^{-2\tau})} \\ &\quad \times \frac{d}{dx} (e^{-x^2/(1-e^{-2\tau})} H_{2n}(x)) \end{aligned} \qquad (3.11b)$$

where the last equality can be verified from the properties of the Hermite polynomials,

$$\frac{d}{dx} H_n(x) = 2n H_{n-1}(x) \quad \text{and} \quad 2x H_j(x) = H_{j+1}(x) + 2j H_{j-1}(x) \qquad (3.12)$$

are skew orthogonal with respect to (3.8).

The structure of p in Proposition 3.1 can be combined with the skew orthogonal property and Dyson's theory of quaternion determinants[10] to give a closed form expression for $\rho_{(n)}$. Let us first recall that a quaternion determinant, denoted Tdet, is defined analogously to an ordinary determinant, as a signed sum over cycles, where the elements in the cycle are quaternions (which can be represented as 2×2 matrices). When the corresponding matrix is self dual, which means that the elements $\mathbf{a}_{jk}$ and $\mathbf{a}_{kj}$ in positions jk and kj respectively are such that if

$$\mathbf{a}_{jk} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad \mathbf{a}_{kj} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

the quaternion determinant is related to the ordinary determinant by

$$\text{Tdet}\, \mathbf{A} = (\det \mathbf{A})^{1/2} \qquad (3.13)$$

On the r.h.s. the matrix $\mathbf{A}$ is that formed when the quaternion elements are regarded as 2×2 blocks to form a matrix with scalar elements.

The distribution function $\rho_{(n)}$ can be written in terms of a quaternion determinant according to the following results.[6]

Proposition 3.2. Suppose N is even. Define the skew-symmetric inner product by (3.7), let $\{U_n(x, \tau)\}_{n=0,1,\dots}$ be a corresponding set of skew orthogonal polynomials with normalization u_n ($U_n(x;\tau)$ and u_n are given explicitly by (3.11) and (3.10) respectively) and set

$$V_n(y;\tau) := \int_{-\infty}^{\infty} dy'\, w_1(y')\, H_a(y,y';\tau)\, U_n(y';\tau)$$

We have

$$\rho_{(n)}(x_1,\ldots,x_n;\tau) = \mathrm{Tdet}\begin{bmatrix} S^{\mathrm{even}}(x_j,x_k;\tau) & I^{\mathrm{even}}(x_j,x_k;\tau) \\ D^{\mathrm{even}}(x_j,x_k;\tau) & S^{\mathrm{even}}(x_k,x_j;\tau) \end{bmatrix}_{j,k=1,\ldots,n}$$

where

$$S^{\mathrm{even}}(x,y;\tau) = \sum_{k=0}^{N/2-1} \frac{w_1(y)}{u_k}(V_{2k}(x;\tau)\, U_{2k+1}(y;\tau) - V_{2k+1}(x;\tau)\, U_{2k}(y;\tau))$$

$$D^{\mathrm{even}}(x,y;\tau) = \sum_{k=0}^{N/2-1} \frac{w_1(x)\, w_1(y)}{u_k}(U_{2k}(x;\tau)\, U_{2k+1}(y;\tau) - U_{2k+1}(x;\tau)\, U_{2k}(y;\tau))$$

$$I^{\mathrm{even}}(x,y;\tau) = \sum_{k=0}^{N/2-1} \frac{1}{u_k}(V_{2k+1}(x;\tau)\, V_{2k}(y;\tau) - V_{2k}(x;\tau)\, V_{2k+1}(y;\tau)) - H_1(x,y;\tau)$$

Proposition 3.3. Suppose N is odd and let $\{U_n(x,\tau)\}_{n=0,1,\ldots}$, $\{V_n(x,\tau)\}_{n=0,1,\ldots}$ and $\{u_n\}_{n=0,\ldots,(N-1)/2-1}$ be as in Proposition 3.2 but redefine $u_{(N-1)/2}$ by

$$u_{(N-1)/2} := a_{(N-1)/2} \qquad \text{where} \quad a_p := \int_{-\infty}^{\infty} dx\, w_1(x)\, F(x)\, U_p(x;\tau)$$

Also introduce the quantities

$$\tilde{U}_n(x;\tau) = U_n(x;\tau) - \frac{a_n}{a_{(N-1)/2}}\, U_{N-1}(x;\tau) \qquad (n=0,\ldots,N-2)$$

$$\tilde{U}_{N-1}(x;\tau) = U_{N-1}(x;\tau)$$

$$\tilde{V}_n(x;\tau) = \int_{-\infty}^{\infty} dy\, w_1(y)\, H_a(x,y;\tau)\, \tilde{U}_n(y;\tau) \qquad (n=0,\ldots,N-1)$$

Then we have

$$\rho_{(n)}(x_1,\dots,x_n;\tau)=\mathrm{Tdet}\begin{bmatrix} S^{\mathrm{odd}}(x,y;\tau) & I^{\mathrm{odd}}(x,y;\tau) \\ D^{\mathrm{odd}}(x,y;\tau) & S^{\mathrm{odd}}(y,x;\tau)\end{bmatrix}_{j,k=1,\dots,n}$$

where

$$\begin{aligned}
S^{\mathrm{odd}}(x,y;\tau) &= \sum_{k=0}^{(N-1)/2-1} \frac{w_1(y)}{u_k}(\tilde{V}_{2k}(x;\tau)\,\tilde{U}_{2k+1}(y;\tau) \\
&\quad -\tilde{V}_{2k+1}(x;\tau)\,\tilde{U}_{2k}(y;\tau))+\frac{1}{u_{(N-1)/2}}F(x;\tau)\,\tilde{U}_{N-1}(y;\tau) \\
D^{\mathrm{odd}}(x,y;\tau) &= \sum_{k=0}^{N/2-1}\frac{w_1(x)\,w_1(y)}{u_k}(\tilde{U}_{2k}(x;\tau)\,\tilde{U}_{2k+1}(y;\tau) \\
&\quad -\tilde{U}_{2k+1}(x;\tau)\,\tilde{U}_{2k}(y;\tau)) \\
I_1^{\mathrm{odd}}(x,y;\tau) &= \sum_{k=0}^{N/2-1}\frac{1}{u_k}(\tilde{V}_{2k+1}(x;\tau)\,\tilde{V}_{2k}(y;\tau) \\
&\quad -\tilde{V}_{2k}(x;\tau)\,\tilde{V}_{2k+1}(y;\tau))-H_1(x,y;\tau) \\
&\quad +\frac{1}{u_{(N-1)/2}}(\tilde{V}_{N-1}(x;\tau)\,F(y;\tau)-F(x;\tau)\,\tilde{V}_{N-1}(y;\tau))
\end{aligned}$$

We stress that these formulas are general in the sense that they still apply whenever g is of the form (2.10b) (of course the skew orthogonal polynomials are dependent on $w_1(x)$).

In the case of specific interest, $w_1(x)=e^{-x^2/2}$, from the explicit formulas (2.17c) and (2.17d) for $U_n(x;\tau)$ and (2.3a) for $g(x,y;\tau)$, use of the orthogonality property of the Hermite polynomials in the definition of V_n in Proposition 3.3 gives

$$V_{2n+1}(x;\tau)=e^{-x^2/2}H_{2n}(x)\,e^{-(2n+1/2)\tau} \tag{3.14a}$$

and

$$\begin{aligned}
V_{2n}(x;\tau) &= e^{-x^2/2}\frac{(2n)!}{2n!}(-1)^n\sum_{k=n}^{\infty}\frac{(-1)^k k!}{(2k+1)!}e^{-(2k+3/2)\tau}H_{2k+1}(x) \\
&= -\frac{e^{-(2n-1/2)\tau}}{1-e^{2\tau}}e^{-x^2/2}e^{x^2/(1-e^{2\tau})}\int_0^x e^{-u^2/(1-e^{2\tau})}H_{2n}(u)\,du
\end{aligned} \tag{3.14b}$$

(the second equality can be verified by multiplying both sides by $e^{x^2/2}e^{-x^2/(1-e^{2\tau})}$ and then differentiating with respect to x; use of the formulas in (3.12) then shows that both sides are equal), while for a_n as defined in Proposition 3.3 and F as defined in Proposition 3.1 we obtain

$$a_n = 0, \quad n \text{ odd} \qquad a_n = H_n(0) = (-1)^{n/2} n!/(n/2)!, \quad n \text{ even} \tag{3.14c}$$

All terms in Propositions 3.2 and 3.3 are therefore known explicitly, and thus the equal parameter distribution for the Gaussian anti-symmetric Hermitian to GUE transition is thereby given.

3.3. Comparison with the Results of Mehta and Pandey

Mehta and Pandey[1] have given quaternion determinant expressions for the equal parameter distributions calculated in the above section. However the expressions of ref. 1 were not derived using the general formulas of Propositions 3.2 and 3.3, and cannot immediately be identified with our results. However, minor manipulation show that the two results do indeed agree, as we will now demonstrate.

Consider for example the expression for the elements $S^{\text{even}}(x, y; \tau)$ in Proposition 3.2 given by Mehta and Pandey:

$$\begin{aligned} S^{\text{even}(MP)}(x, y; \tau) = {} & \sum_{k=0}^{N-1} \frac{e^{-(x^2+y^2)/2}}{\sqrt{\pi}\, 2^k k!} H_k(x)\, H_k(y) \\ & - \frac{e^{-(x^2+y^2)/2}}{\sqrt{\pi}\, 2^{N-1}(N/2-1)!} (-1)^{N/2} e^{(N-1/2)\tau} H_{N-1}(y) \\ & \times \sum_{k=N/2}^{\infty} \frac{(-1)^k k!}{(2k+1)!} e^{-(2k+3/2)\tau} H_{2k+1}(y) \end{aligned} \tag{3.15}$$

(in the final term we have written a sum whereas in ref. 1 it is given as an integral; that the two expressions are identical follows by use of the formulas (3.14b). On the other hand, from Proposition 3.2, (3.11) and (3.13) our expression is

$$\begin{aligned} S^{\text{even}}(x, y; \tau) = {} & \sum_{\substack{k=0 \\ k \text{ even}}}^{N-1} \frac{e^{-(x^2+y^2)/2}}{\sqrt{\pi}\, 2^k k!} H_k(x)\, H_k(y) \\ & + e^{-(x^2+y^2)/2} \sum_{n=N/2}^{\infty} \frac{(-1)^n}{\pi^{1/2} 2^{2n+1} n!} (e^{(2n+3/2)\tau} H_{2n+1}(y) \\ & + 4n e^{(2n-1/2)\tau} H_{2n-1}(y)) \sum_{k=n}^{\infty} \frac{(-1)^k k!}{(2k+1)!} e^{-(2k+3/2)\tau} H_{2k+1}(x) \end{aligned} \tag{3.16}$$

We observe that the double summation has the structure

$$\sum_{n=0}^{N/2-1} (\alpha_{2n+1} - \alpha_{2n-1}) \sum_{k=n}^{\infty} b_{2k+1}$$

with $\alpha_1 := 0$, which is equal to

$$\sum_{n=0}^{N/2-1} \alpha_{2n+1} b_{2n+1} + \alpha_{N-1} \sum_{k=N/2}^{\infty} b_{2k+1}$$

The agreement between (3.15) and (3.16) now follows.

The agreement between the other elements in the quaternion determinant of Proposition 3.2 (and the elements in the quaternion determinant of Proposition 3.3) as calculated from the formulas of this paper and those given by Mehta and Pandey can be demonstrated by similar straightforward manipulation.

3.4. The Scaled $N \to \infty$ Limit for N Even

To compute the scaled $N \to \infty$ limit (N even) of ρ_n, we found it convenient to use the expression (3.15) for $S^{\text{even}}(x, y; \tau)$ and the expressions for I^{even} and D^{even} as given in Proposition 3.2. As in Section 2.3, our method is to compute the asymptotic behavior of the summands, which requires the asymptotics of $U_n(x; \tau)$ and $V_n(x; \tau)$. From the asymptotic formula for $H_n(x)$ noted below (2.19) and the formulas (3.11) and (3.13) we find that in the scaled limit (2.20), with $n/N := t$ fixed

$$\frac{w_1(x)\, U_{2n}(x; \tau)}{u_n^{1/2}} \sim (-1)^n \left(\frac{2}{N}\right)^{1/4} \frac{e^{(\pi\rho)^2 t\tau/2}}{(\pi^2 t)^{1/4}} \cos \pi\rho x \sqrt{t} \tag{3.17a}$$

$$\frac{w_1(x)\, U_{2n+1}(x; \tau)}{u_n^{1/2}} \sim (-1)^n \frac{4}{N\pi^{1/2}} \left(\frac{N}{2}\right)^{1/4} t^{1/4} \frac{d}{dt}$$

$$\times (\sin \pi\rho x \sqrt{t}\, e^{(\pi\rho)^2 t\tau/2}) \tag{3.17b}$$

$$\frac{w_1(x)\, V_{2n}(x; \tau)}{u_n^{1/2}} \sim \frac{(-1)^{n+1}}{2(\pi^2 n)^{1/4}} \left(\frac{N}{2}\right)^{1/2} \int_t^{\infty} ds \frac{\sin \pi\rho \sqrt{s}}{\sqrt{s}} e^{-(\pi\rho)^2 s\tau/2} \tag{3.17c}$$

$$\frac{w_1(x)\, V_{2n+1}(x; \tau)}{u_n^{1/2}} \sim (-1)^n \left(\frac{2}{N}\right)^{1/4} \frac{e^{-(\pi\rho)^2 t\tau/2}}{(\pi^2 t)^{1/4}} \cos \pi\rho x \sqrt{t} \tag{3.17d}$$

For the quantity I^{even}, also required is the asymptotic behavior of $H_a(x, y; \tau)$. Now use of the asymptotic formula (2.20e) in the definition of H_a (recall Proposition 3.1) and interchange of the order of integrations allows the integral over u to be carried out. This gives

$$H_a(x, y; \tau) \sim \frac{N}{\pi} \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \, e^{-2(\pi\rho)^2 (v_1^2 + v_2^2)} \times \operatorname{sgn}(v_1 + v_2)(\sin 2\pi v_1 \rho x \cos 2\pi v_2 \rho y - (x \leftrightarrow y)) \quad (3.18a)$$

which after the change of variables $u_1 = v_1 + v_2$, $u_2 = v_1 - v_2$ reduces to

$$H_a(x, y; \tau) \sim \frac{16N}{\pi} \int_0^{\infty} dt \, e^{-4(\pi\rho)^2 \tau t^2} \cos 2\pi\rho(x + y)\, t \times \int_0^{\infty} ds \, e^{-4(\pi\rho)^2 \tau s^2} \sin 2\pi\rho(x - y)\, s \quad (3.18b)$$

Substituting the asymptotic formulas (3.17) and (3.18b) in the expression (3.15) for S^{even} and the expressions for D^{even} and I^{even} in Proposition 3.2, we see that to leading order the sums become Riemann integrals. Explicitly, after simplification we find

$$S^{\text{even}}(x, y; \tau) \sim \frac{(2N)^{1/2}}{\pi\rho} \hat{S}^{\text{even}}(x, y; \tau) \quad (3.19a)$$

$$D^{\text{even}}(x, y; \tau) \sim -\frac{1}{2\pi\rho} \hat{D}^{\text{even}}(x, y; \tau) \quad (3.19b)$$

$$I^{\text{even}}(x, y; \tau) \sim -\frac{4N}{\pi\rho} \hat{I}^{\text{even}}(x, y; \tau) \quad (3.19c)$$

where

$$\hat{S}^{\text{even}}(x, y; \tau) = \rho \left(\frac{\sin \pi\rho(y - x)}{\pi\rho(y - x)} + 2e^{(\pi\rho)^2 \tau/2} \sin \pi\rho y \times \int_{1/2}^{\infty} e^{-2(\pi\rho)^2 \tau s^2} \sin 2\pi s\rho x \, ds \right) \quad (3.20a)$$

$$\hat{D}^{\text{even}}(x, y; \tau) = \rho \left(4\pi\rho(y - x) \int_0^{1/2} e^{4(\pi\rho)^2 t^2 \tau} \cos 2\pi\rho(x + y)\, t \, dt + 2e^{(\pi\rho)^2 \tau} \sin \pi\rho(y - x) \right) \quad (3.20b)$$

$$\tilde{I}^{\text{even}}(x, y; \tau) = \rho \left(\int_0^{1/2} dt\, e^{-2(\pi\rho)^2 t^2 \tau} \cos 2\pi\rho y t \right.$$
$$\left. \times \int_t^{\infty} ds\, e^{-2(\pi\rho)^2 s^2 \tau} \sin 2\pi s \rho x - (x \leftrightarrow y) \right)$$
$$+ 4\rho \int_0^{\infty} du_1\, e^{-4(\pi\rho)^2 u_1^2 \tau} \sin 2\pi\rho u_1 (x - y)$$
$$\times \int_0^{\infty} du_2\, e^{-4(\pi\rho)^2 u_2^2 \tau} \cos 2\pi\rho u_2 (x + y) \qquad (3.20c)$$

and consequently

$$\lim_{N \to \infty} \left(\frac{\pi\rho}{\sqrt{2N}} \right)^n \rho_n(\pi\rho x_1/\sqrt{2N}, \ldots, \pi\rho x_n/\sqrt{2N}; (\pi\rho)^2 \tau/2N)$$
$$= \text{Tdet} \begin{bmatrix} \tilde{S}^{\text{even}}(x_j, x_k; \tau) & \tilde{I}^{\text{even}}(x_j, x_k; \tau) \\ \tilde{D}^{\text{even}}(x_j, x_k; \tau) & \tilde{S}^{\text{even}}(x_k, x_j; \tau) \end{bmatrix}_{j,k=1,\ldots,n} \qquad (3.21)$$

3.5. The Scaled $N \to \infty$ Limit for N Odd

Comparison of Propositions 3.2 and 3.3 shows the quantities S^{odd} etc. to be closely related to S^{even} etc.. This allows use to be made of the results of the previous section in calculating the asymptotics of the sums in Proposition 3.3 for N odd.

For example the sum defining D^{odd} in Proposition 3.3 can be written

$$D^{\text{odd}}(x, y; \tau) := \sum_{k=0}^{(N-1)/2-1} \frac{w_1(x)\, w_1(y)}{u_k} (\tilde{U}_{2k}(x; \tau)\, \tilde{U}_{2k+1}(y; \tau) - (x \leftrightarrow y))$$
$$= \sum_{k=0}^{(N-1)/2-1} \frac{w_1(x)\, w_1(y)}{u_k} (U_{2k}(x; \tau)\, U_{2k+1}(y; \tau) - (x \leftrightarrow y))$$
$$- \left[\sum_{k=0}^{(N-1)/2-1} \frac{w_1(x)\, w_1(y)}{u_k} \right.$$
$$\left. \times \left(\frac{a_k}{a_{(N-1)/2}} \right) U_{2k+1}(y; \tau)\, U_{N-1}(x; \tau) - (x \leftrightarrow y) \right] \qquad (3.22)$$

The first sum of the second equality is essentially D^{even} (the upper terminals are slightly different, but this makes no difference to the scaled $N \to \infty$

limit), and using (2.17b), (3.14c) and (3.17) the leading asymptotics of the final term is readily computed. Combining the results we find

$$D^{\mathrm{odd}}(x, y; \tau) \sim -\frac{1}{2\pi\rho} \hat{D}^{\mathrm{odd}}(x, y; \tau) \tag{3.23a}$$

where

$$\hat{D}^{\mathrm{odd}}(x, y; \tau) = \rho \left(4\pi\rho(y-x) \int_0^{1/2} e^{4(\pi\rho)^2 t^2 \tau} \cos 2\pi\rho(x+y)\, t\, dt - 2e^{(\pi\rho)^2 \tau} \sin \pi\rho(y-x) \right) \tag{3.23b}$$

Proceeding similarly, we also find

$$S^{\mathrm{odd}}(x, y; \tau) \sim \frac{(2N)^{1/2}}{\pi\rho} \hat{S}^{\mathrm{odd}}(x, y; \tau) \tag{3.24a}$$

$$I^{\mathrm{odd}}(x, y; \tau) \sim -\frac{4N}{\pi\rho} \hat{I}^{\mathrm{odd}}(x, y; \tau) \tag{3.25a}$$

where

$$\hat{S}^{\mathrm{odd}}(x, y; \tau) = \rho \left(\frac{\sin \pi\rho(y-x)}{\pi\rho(y-x)} + 2e^{(\pi\rho)^2 \tau/2} \cos \pi\rho y \times \int_{1/2}^{\infty} ds\, e^{-2(\pi\rho)^2 \tau s^2} \cos 2\pi s\rho x \right) \tag{3.24b}$$

$$\begin{aligned}\hat{I}^{\mathrm{odd}}(x, y; \tau) = {} & \rho \left(\int_0^{1/2} dt\, e^{-2(\pi\rho)^2 t^2 \tau} \cos 2\pi\rho y t \times \int_t^{\infty} ds\, e^{-2(\pi\rho)^2 s^2 \tau} \sin 2\pi s\rho x - (x \leftrightarrow y) \right) \\ & + \rho \left(\int_{1/2}^{\infty} dt\, e^{-2(\pi\rho)^2 t^2 \tau} \cos 2\pi\rho y t \times \int_{1/2}^{\infty} ds\, e^{-2(\pi\rho)^2 s^2 \tau} \sin 2\pi s\rho x - (x \leftrightarrow y) \right) \\ & + 4\rho \int_0^{\infty} du_1\, e^{-4(\pi\rho)^2 u_1^2 \tau} \sin 2\pi\rho u_1(x-y) \times \int_0^{\infty} du_2\, e^{-4(\pi\rho)^2 u_2^2 \tau} \cos 2\pi\rho u_2(x+y)\end{aligned} \tag{3.25b}$$

and consequently (3.21) applies for the limiting value of ρ_n, with $\hat{S}^{\text{even}}$ etc. replaced by $\hat{S}^{\text{odd}}$ etc.

As with the density–density correlation of Section 2.3, it is possible to empirically calculate the one and two point equal-parameter distributions of this section using data from computer generated Gaussian random matrices. For definiteness suppose N is odd. Now in the scaled $N \to \infty$ limit, (3.21) modified so that the superscripts "even" are replaced by "odd" and the formula (3.13) for the computation of the quaternion determinant gives

$$\rho_{(1)}(x; \tau) = \hat{S}^{\text{odd}}(x, x; \tau) \tag{3.26a}$$

and

$$\begin{aligned}\rho_{(2)}(x, y; \tau) = \hat{S}^{\text{odd}}(x, x; \tau)\, \hat{S}^{\text{odd}}(y, y; \tau) - (\hat{S}^{\text{odd}}(x, y; \tau)\, \hat{S}^{\text{odd}}(y, x; \tau) \\ - \hat{D}^{\text{odd}}(x, y; \tau)\, \hat{I}^{\text{odd}}(x, y; \tau))\end{aligned} \tag{3.26b}$$

where $\hat{S}^{\text{odd}}$ is given by (3.24b), $\hat{D}^{\text{odd}}$ is given by (3.23b) and $\hat{I}^{\text{odd}}$ is given by (3.25b).

On the other hand, these distributions are the scaled limit of the corresponding distributions for parameter-dependent Gaussian random matrices with elements distributed according to (3.2). To calculate $\rho_{(1)}(x; \tau)$ for these random matrices we choose a definite value of N ($N = 11$) and use a computer program to generate a large number M ($M = 5000$) of random matrices specified according to (3.2) with a definite value of τ ($\tau = 0.015$). The eigenvalues of each matrix are calculated and multiplied by the scale $\pi/(2N - (\pi x)^2)^{1/2}$ so that the density at large distances from the origin is unity. The number of scaled eigenvalues in intervals of length dX ($dX = 0.1$) is calculated and divided by M to give an empirical bar graph of the density. The bar graph is plotted in Fig. 2a and compared with the theoretical curve specified by (3.26a), with $\rho = 1$ and τ replaced by $2N\tau/\pi^2$.

The two-point equal-parameter distribution can be calculated empirically from a formula analogous of (2.24):

$$\frac{\rho_{(2)}(X, Y; \tau)}{\rho_{(1)}(Y; \tau)} = \sum{}' P(j; X, Y) \tag{3.27}$$

where $P(j; X, Y)$ denotes the p.d.f. for the event that the scaled eigenvalue with label j occurs within the interval $[X, X + dX]$, given that there is a scaled eigenvalue at the point Y. The dash on the sum indicates that the label of the scaled eigenvalue at Y is not included in the sum. To use this

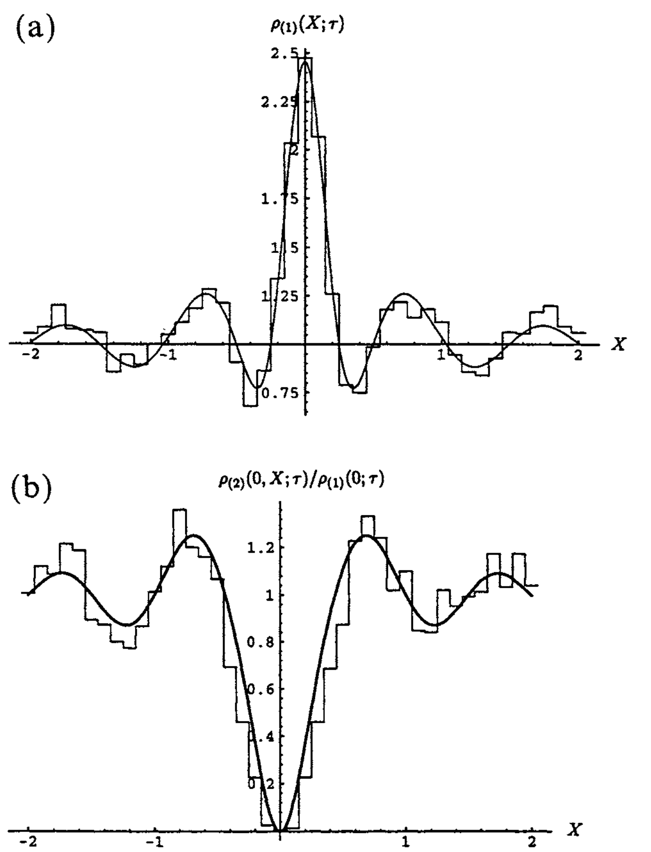

Fig. 2. (a) Comparison between the density $\rho_{(1)}(X;\tau)$, $\tau=0.015$, calculated empirically from 5,000 computer generated parameter-dependent Gaussian random matrices, intermediate between anti-symmetric Hermitian and Hermitian and of dimension 11×11, and the scaled limit of the same quantity. (b) Same as in Fig. 2a for the quantity ${}_d\rho^T_{(2)}((0,0),(X;\tau))/\rho_{(1)}(0;0)$ with $\tau=0.015$. The bar graph was generated from 3,000 pairs of random matrices of dimension 12×12.

formula, we first proceed as in the calculation of the density and calculate and scale the eigenvalues of each computer-generated random matrix (we chose $\tau=0.015$). Unlike the calculation of the density, it is necessary to reject a large number of these matrices because there will be no scaled eigenvalue is some small neighborhood $[-dY/2+Y, Y+dY/2]$ of the point Y. We chose $Y=0$ and $dY=0.2$. A total of 1500 11×11 accepted random matrices were generated, and the eigenvalues recorded for the empirical computation of the p.d.f.'s $P(j;X,0)$ and thus from (3.27) for the empirical computation of two-point equal-parameter distribution. The results of this calculation are given as a bar graph in Fig. 2b and compared with the theoretical curve specified by (3.26b) with $\rho = 1$ and τ replaced by $2N\tau/\pi^2$.

4. CIRCULAR ENSEMBLE DESCRIPTION

A Brownian motion theory of parameter-dependent circular ensembles was devised by Dyson.[3] The theory gives that the eigenvalue p.d.f. of a random unitary matrix evolving towards a member of the CUE satisfies the Fokker–Planck equation in Proposition 2.1 with W therein replaced by its periodic version:

$$W = -\sum_{1 \leqslant j < k \leqslant N} \log |\sin \pi(x_k - x_j)/L|, \qquad -L/2 \leqslant x_j \leqslant L/2 \tag{4.1}$$

For the COE → CUE transition the density–density correlation has been calculated in ref. 5 (see eq. (1.1) above) and the equal-parameter distribution has been calculated in ref. 11. For both quantities, agreement was found with the corresponding quantities in the scaled $N \to \infty$ limit of the GOE → GUE transition (for the density–density correlation, this was demonstrated in Section 2.3).

Our objective in this section is to calculate the equal parameter distribution in the circular ensemble for the analogue of the Gaussian antisymmetric Hermitian to GUE transition. The analogue of the eigenvalue p.d.f. is defined as the solution of the Fokker–Planck equation in Proposition 2.1 with W given by (4.1) and initial conditions

$$\begin{aligned} f(x_1,\dots, x_N) = \frac{1}{C_N} \prod_{1 \leqslant j < k \leqslant [N/2]} & \sin \pi(x_j - x_k)/L \sin \pi(x_j + x_k)/L \\ & \times \prod_{j=1}^{[N/2]} \delta(x_j - x_{N+1-j})\, \chi_N^{(C)}(x_1,\dots, x_{(N+1)/2}) \end{aligned} \tag{4.2a}$$

where

$$\chi_N^{(C)} = 1 \quad \text{for } N \text{ even}, \qquad \chi = \prod_{j=1}^{[N/2]} \sin^2 \pi x_j/L\, \delta(x_{(N+1)/2}) \quad \text{for } N \text{ odd} \tag{4.2b}$$

Since the Fokker–Planck equation in Proposition 2.1 with W replaced by (4.1) has Green function[5,10]

$$G(x_1^{(0)},..., x_N^{(0)} \,|\, x_1,..., x_N; \tau) = e^{\tau E_0} \prod_{1 \leqslant j<k \leqslant N} \frac{\sin \pi(x_k - x_j)/L}{\sin \pi(x_k^{(0)} - x_j^{(0)})/L} \det[g^{(C)}(x_j^{(0)}, x_k)]_{j,k=1,...,N} \tag{4.3a}$$

where

$$\begin{gathered} g^{(C)}(x_j^{(0)}, x_k) = \frac{1}{L} \theta_s(\pi(x_j^{(0)} - x_k)/L; q) \\ E_0 := \left(\frac{2\pi}{L}\right)^2 \frac{N(N^2-1)}{12} \qquad \text{and} \qquad q := e^{-2\pi^2\tau/L^2} \end{gathered} \tag{4.3b}$$

with $\theta_s = \theta_2$ for N even, $\theta_s = \theta_3$ for N odd, our task is to compute (3.4) and (3.5) with G given by (4.3a) and f given by (4.2).

Our strategy is analogous to that used in the previous section: we first express the eigenvalue p.d.f. (3.4) as a Pfaffian and then write ρ_n as a quaternion determinant involving appropriate skew orthogonal polynomials. With G given by (4.3) and f given by (4.2), the eigenvalue p.d.f. (3.4) is given in terms of a Pfaffian according to Proposition 3.1 with H_a replaced by

$$\begin{aligned} H_a^{(C)}(x, y; \tau) := \frac{1}{2L^2} \int_0^{L/2} \frac{du}{\sin 2\pi u/L} (&\theta_s(\pi(u-x)/L; q)\, \theta_s(\pi(u+y)/L; q) \\ &- \theta_s(\pi(u-y)/L; q)\, \theta_s(\pi(u+x)/L; q)) \end{aligned} \tag{4.4a}$$

and $F(x; \tau)$ replaced by

$$F^{(C)}(x; \tau) := \tfrac{1}{2}\theta_s(\pi x/L; q) \tag{4.4b}$$

(of course E_0 and C_N also require replacing).

The appropriate skew symmetric inner product $\langle\ \rangle_a^{(\tau)(C)}$ say is now given by (c.f. (3.7))

$$\langle f \,|\, g\rangle_a^{(\tau)(C)} = \int_{-L/2}^{L/2} dy_1 \int_{-L/2}^{L/2} dy_2\, H_a^{(C)}(y_1, y_2; \tau)\, f(y_1)\, g(y_2) \tag{4.5a}$$

and in the limit $\tau \to 0$ this reduces to

$$\langle f \,|\, g\rangle_a^{(0)(C)} = \frac{1}{2} \int_0^{L/2} \frac{dx}{\sin 2\pi x/L} (f(x)\, g(-x) - f(-x)\, g(x)) \tag{4.5b}$$

For the circular ensemble the skew orthogonal "polynomials" are linear combinations from the set $\{e^{2\pi i(j-(N+1)/2)x/L}\}_{j=1,\dots,N}$. This is a consequence of the Vandermonde-type expansion

$$\prod_{1\leqslant j<k\leqslant N} 2\sin\pi(x_k-x_j)/L = i^{-N(n-1)/2}\det[e^{2\pi i x_j(k-(N+1)/2)/L}]_{j,k=1,\dots,N}$$

used in the Green function (4.3a). From the theory of Section 2.2 we have that if the skew orthogonal polynomials of the $\tau=0$ inner product are of the form

$$U_n(x;0)=\sum_{n=1}^{N} a_n e^{2\pi i(n-(N+1)/2)x/L} \tag{4.6a}$$

then the skew orthogonal polynomials for the inner product (4.5a) are given by

$$U_n(x;\tau)=\sum_{n=1}^{N} a_n q^{(n-(N+1)/2))^2 x/L}\, e^{2\pi i(n-(N+1)/2)x/L} \tag{4.6b}$$

Using the integration formula

$$\int_0^{L/2}\frac{\sin 2\pi p x/L}{\sin 2\pi x/L}\,dx=\begin{cases}(L/2)\operatorname{sgn}(p), & p \text{ odd}\\ 0, & p \text{ even}\end{cases} \tag{4.7}$$

we can readily check that the functions

$$U_n(x;0)=\begin{cases} e^{2\pi i(n+1-(N+1)/2)x/L}+e^{2\pi i(1-(N+1)/2)x/L} & n \text{ even}\\ e^{2\pi i(n+1-(N+1)/2)x/L}-e^{2\pi i(n-1-(N+1)/2)x/L} & n \text{ odd}\neq 1\\ e^{2\pi i(2-(N+1)/2)x/L} & n=1\end{cases} \tag{4.8a}$$

are skew orthogonal with respect to (4.5b) with normalization

$$u_n=-iL \tag{4.8b}$$

and are of the form (4.6a). Thus, according to (4.6b),

$$U_n(x,\tau)=\begin{cases} q^{-(n+1-(N+1)/2)^2}e^{2\pi i(n+1-(N+1)/2)x/L} & \\ \quad +q^{-(1-(N+1)/2)^2}e^{2\pi i(1-(N+1)/2)x/L} & n \text{ even}\\ q^{-(n+1-(N+1)/2)^2}e^{2\pi i(n+1-(N+1)/2)x/L} & \\ \quad -q^{-(n-1-(N+1)/2)^2}e^{2\pi i(n-1-(N+1)/2)x/L} & n \text{ odd}\neq 1\\ q^{-(2-(N+1)/2)^2}e^{2\pi i(2-(N+1)/2)x/L} & n=1\end{cases} \tag{4.8c}$$

are skew orthogonal with respect to (4.5a).

Propositions 3.2 and 3.3 still apply for ρ_n, except that now $w_1(y)=1$, H_a is to be replaced by $H_a^{(C)}$ as given by (4.4a), $F(x;\tau)$ is to be replaced by $F^{(C)}(x;\tau)$ as given by (4.4b) and the intervals of integration $(-\infty,\infty)$ are now $[-L/2, L/2]$. Using (4.7) to carry out the integrations in the modified form of the definition of $V_k(x;\tau)$ given in Proposition 3.2, we obtain the explicit formulas

$$V_{2n-1}(x;\tau)=iq^{(3/2-2n+N/2)^2}e^{2\pi i(3/2-2n+N/2)\,x/L} \qquad (n\neq 1) \tag{4.9a}$$

$$V_{2n-2}(x;\tau)=\frac{i}{2}\left(\sum_{p=-\infty}^{\infty}\operatorname{sgn}(p-1/2)\,q^{(2p-2n+(N+1)/2)^2}e^{2\pi i(2p-2n+(N+1)/2)\,x/L}\right.$$
$$\left.+\sum_{p=-\infty}^{\infty}\operatorname{sgn}(p-1/2)\,q^{(2p-2+(N+1)/2)^2}e^{2\pi i(2p-2+(N+1)/2)\,x/L}\right) \tag{4.9b}$$

$$V_1(x;\tau)=\frac{i}{2}\sum_{p=-\infty}^{\infty}\operatorname{sgn}(p-1/2)\,q^{(2p-3+(N+1)/2)^2}e^{2\pi i(2p-3+(N+1)/2)\,x/L} \tag{4.9c}$$

while carrying out the integration in the modified form of the definition of a_j in Proposition 3.3 gives

$$a_j=\begin{cases}0, & j \text{ odd } (\neq 1)\\ 1, & j=1\\ 2, & j \text{ even}\end{cases} \tag{4.10}$$

All the quantities in Propositions 3.2 and 3.3 are thus known explicitly, and therefore so is ρ_n in the finite system. To evaluate ρ_n in the thermodynamic limit, it still remains to obtain the asymptotic values of $\{U_n(x;\tau)\}_{n=0,1,\dots}$ etc. which, following the strategy used in the previous sections, allows the asymptotic value of the sums to be computed. The cases N even and N odd will be treated separately.

***N* even**

From (4.8) and (4.9), for $N\to\infty$ and $2j/N=\,=:t$ fixed,

$$U_{2j}(x;\tau)\sim e^{2(\pi\rho)^2\tau(t-1/2)^2}e^{2\pi i\rho x(t-1/2)}+e^{(\pi\rho)^2\tau/2}e^{-\pi i\rho x} \tag{4.11a}$$

$$U_{2j+1}(x;\tau)\sim\frac{2}{N}\frac{d}{dt}\left(e^{2(\pi\rho)^2\tau(t-1/2)^2}e^{2\pi i\rho x(t-1/2)}\right) \tag{4.11b}$$

$$V_{2j+1}(x;\tau)\sim ie^{-2(\pi\rho)^2\tau(t-1/2)^2}e^{-2\pi i\rho x(t-1/2)} \tag{4.11c}$$

$$V_{2j}(x;\tau)\sim\frac{iN}{4}\left(\int_{-\infty}^{\infty}ds\,\mathrm{sgn}(s)\,e^{-2(\pi\rho)^2\tau(s-t+1/2)^2}e^{2\pi i\rho x(s-t+1/2)}\right.$$
$$\left.+\int_{-\infty}^{\infty}ds\,\mathrm{sgn}(s)\,e^{-2(\pi\rho)^2\tau(s+1/2)^2}e^{2\pi i\rho x(s+1/2)}\right)\tag{4.11d}$$

Also[5]

$$g^{(C)}(x,y;\tau)\sim\left(\frac{1}{2\pi\tau}\right)^{1/2}\exp[-(x-y)^2/2\tau]\tag{4.12a}$$

so comparison with (2.20e) shows

$$H_a^{(C)}(x,y;\tau)\sim\frac{\pi\rho}{4}H_a(x,y;\tau)\tag{4.12b}$$

Using these results we see that, as in the Gaussian case, the sums in Proposition 3.2 tend to integrals in the thermodynamic limit. Straightforward manipulation of the integrals gives

$$S^{\mathrm{even}(C)}(x;y;\tau)\sim\hat{S}^{\mathrm{even}}(x,y;\tau)\tag{4.13a}$$

$$D^{\mathrm{even}(C)}(x,y;\tau)\sim-\frac{1}{N}\hat{D}^{\mathrm{even}}(x,y;\tau)\tag{4.13b}$$

$$I^{\mathrm{even}(C)}(x,y;\tau)\sim-N\hat{I}^{\mathrm{even}}(x,y;\tau)\tag{4.13c}$$

where $\hat{S}^{\mathrm{even}}$ etc. are given by (3.20). In the thermodynamic limit ρ_n is therefore given by the right hand side of (3.21), in agreement with the scaled limit in the Gaussian case.

N **odd**

From (4.8) and (4.9), modified according to the prescription of Proposition 3.3, for $N\to\infty$ with N odd and $2j/N=t$ fixed,

$$\tilde{U}_{2j}(x;\tau)\sim e^{2(\pi\rho)^2\tau(t-1/2)^2}e^{2\pi i\rho x(t-1/2)}-e^{(\pi\rho)^2\tau/2}e^{\pi i\rho x}\tag{4.14a}$$

$$\tilde{U}_{2j+1}(x;\tau)\sim\frac{2}{N}\frac{d}{dt}\left(e^{2(\pi\rho)^2\tau(t-1/2)^2}e^{2\pi i\rho x(t-1/2)}\right)\tag{4.14b}$$

$$\tilde{V}_{2j+1}(x;\tau)\sim ie^{-2(\pi\rho)^2\tau(t-1/2)^2}e^{-2\pi i\rho x(t-1/2)})\tag{4.14c}$$

$$\tilde{V}_{2j}(x;\tau)\sim\frac{iN}{4}\left(\int_{-\infty}^{\infty}ds\,\mathrm{sgn}(s)\,e^{-2(\pi\rho)^2\tau(s-t+1/2)^2}e^{2\pi i\rho x(s-t+1/2)}\right.$$
$$\left.-\int_{-\infty}^{\infty}ds\,\mathrm{sgn}(s)\,e^{-2(\pi\rho)^2\tau(s-1/2)^2}e^{2\pi i\rho x(s-1/2)}\right)\tag{4.14d}$$

and the result (4.12b) holds independent of the parity of N. Substitution in Proposition 3.3 shows that the results (4.13) again hold, with the superscript "even" now replaced by "odd" ($\hat{S}^{\rm odd}$ etc. is given by (4.23b)–(4.25b)), and consequently the thermodynamic value of ρ_n with N odd agree with the result obtained in the scaled $N \to \infty$ limit of the Gaussian case, which is given by the right hand side of (3.21) with $\hat{S}^{\rm even}$ etc. replaced by $\hat{S}^{\rm odd}$ etc..

5. DENSITY–DENSITY CORRELATION FOR THE CIRCULAR ENSEMBLE DESCRIPTION

The formalism of Section 2 only requires minor modification to be applicable to the calculation of the density–density correlation in the circular ensemble Brownian motion model of the previous section. This is also true of the calculation of the density–density correlation for the Gaussian anti-symmetric Hermitian to GUE transition. However, in the latter case, it turns out that the task of simplifying the analogue of the sums in Proposition 2.3 in the scaled $N \to \infty$ limit is very tedious. From the evidence of the results of the calculations for ρ_n, we would expect both descriptions would give the same expressions in the $N \to \infty$ limit, so little is lost by omitting this calculation. Even in the circular ensemble case, the task of simplifying the analogue of the sums in Proposition 2.3 is still difficult; for this reason we have restricted attention to the calculation of ${}_d\rho^T_{(2)}((x_a, \tau_a), (x_b, \tau_b))$ with $\tau_a = 0$.

As we have noted, the formula in Proposition 2.3 for ${}_d\rho^T_{(2)}$ is again applicable in the present setting, after minor modification. These modifications are necessary since in the derivation of Proposition 2.3 it was assumed that the initial distribution was of the form (2.12) and that N was even. Also, we want to specialize the formulas so that $\tau_a = 0$. Taking these points of detail into consideration, but still following the general strategy which led to Proposition 2.3, we find that the analogue of Proposition 2.3 in the present setting is given by the following result.

Proposition 5.1. Let N be even and suppose the initial distribution is given by (4.2) and the Green function is given by (4.3). Define

$$W_{jk} := \int_{-L/2}^{L/2} \frac{a(y)}{\sin 2\pi y/L} (A_j(y)\, A_k(-y) - A_j(-y)\, A_k(y))\, dy$$

where

$$A_j(y) := \int_{-L/2}^{L/2} b(s)\, g^{(C)}(y, s; \tau)\, U_j(s; \tau)\, ds$$

with $\{U_j(x;\tau)\}_{j=1,...}$ given by (4.8c). Then ${}_d\rho^T_{(2)}((x_a,0),(x_b,\tau_b))$ is given by the formula given in Proposition 2.3 for ${}_d\rho^T_{(2)}$ with h_j replaced by u_j and $\tau_a=0$, $\tau_b=\tau$. For N odd modify the definition of $A_j(y)$ to read

$$A_j(y):=\int_{-L/2}^{L/2} b(s)\, g^{(C)}(y,s;\tau)\, \tilde{U}_j(s;\tau)\, ds$$

and let the above form for W_{jk} apply for $k(j)\neq N+1$, while for $k(j)=N+1$ let

$$W_{j,N+1}=-W_{N+1,j}:=\tfrac{1}{2}\int_{-L/2}^{L/2} b(s)\, g^{(C)}(y,s;\tau)\, \tilde{U}_j(s;\tau)\, ds$$

Then again ${}_d\rho^T_{(2)}((x_a,0),(x_b,\tau_b))$ is given by the formula given in Proposition 2.3 for ${}_d\rho^T_{(2)}$ with h_j replaced by u_j and $\tau_a=0$, $\tau_b=\tau$ and the upper terminals of summation equal to $(N+1)/2$.

The task is now to compute the above sums in Proposition 2.3, modified according to the above proposition, in the thermodynamic limit. As we have done throughout, we do this by computing the asymptotics of the summands. The sums then become Riemann integrals, which we attempt to express in the simplest form. As the calculation is very similar to that presented in previous sections, we will present only the result for the asymptotics of the distinct sums in Proposition 2.3.

***N* even**

For N even we find

$$\begin{aligned}\sum_{j=1}^{N/2} &\frac{K_{2j,2j-1}(x_a,x_b;\tau)}{u_j}\\ &\sim 4\rho^2\left(\int_0^{1/2} dt\, e^{2(\pi\rho)^2t^2\tau}\cos 2\pi\rho t(x_a-x_b)\right.\\ &\quad\times\int_0^{\infty} du\, e^{-2(\pi\rho)^2u^2\tau}\cos 2\pi\rho u(x_a-x_b)\\ &\quad+\int_0^{1/2} dt\, e^{2(\pi\rho)^2t^2\tau}\cos 2\pi\rho t(x_a+x_b)\\ &\quad\left.\times\int_0^{\infty} du\, e^{-2(\pi\rho)^2u^2\tau}\cos 2\pi\rho u(x_a+x_b)\right)\end{aligned}\tag{5.1a}$$

$$\sum_{j,j'=2}^{N/2} \frac{1}{u_j u_{j'}} I_{2j,2j'-1}(x_a;0)\, J_{2j',2j-1}(x_b;\tau)$$

$$\sim -\frac{\rho^2}{4}\Bigg(-\frac{16}{2\pi\rho x_a} e^{(\pi\rho)^2\tau/2} \sin 2\pi\rho x_a \sin \pi\rho x_b$$

$$\times \int_0^\infty du\, e^{-2(\pi\rho)^2 u^2\tau} \sin 2\pi\rho u x_b + \frac{1}{2\pi\rho x_a} e^{(\pi\rho)^2\tau/2} \sin 2\pi\rho x_a$$

$$\times \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau}\, 2\,\mathrm{Re}\big(-6 e^{2\pi i\rho x_b(u+1/2)} + 2e^{2\pi i\rho x_b(u-1/2)}\big)$$

$$-16 \sin \pi\rho x_a \int_0^{1/2} du\, e^{2(\pi\rho)^2 u^2\tau} \sin 2\pi\rho u x_b \sin 2\pi\rho u x_a$$

$$\times \int_{1/2}^\infty dt\, e^{-2(\pi\rho)^2 t^2\tau} \sin 2\pi\rho t x_b - \frac{16}{2\pi\rho x_a} e^{(\pi\rho)^2\tau/2} \cos \pi\rho x_a \sin \pi\rho x_b$$

$$\times \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau} \sin 2\pi\rho u x_b \sin 2\pi\rho u x_a$$

$$-8 \int_0^{1/2} dt\, e^{2(\pi\rho)^2 t^2\tau} \cos 2\pi\rho t(x_a - x_b)$$

$$\times \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u(x_a - x_b)$$

$$-8 \int_0^{1/2} dt\, e^{2(\pi\rho)^2 t^2\tau} \cos 2\pi\rho t(x_a + x_b)$$

$$\times \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u(x_a + x_b)\Bigg) \qquad (5.1b)$$

$$\sum_{j,j'=2}^{N/2} \frac{1}{u_1 u_{j'}} I_{2,2j'-1}(x_a;0)\, J_{2j',1}(x_b;\tau) + \sum_{j=2}^{N/2} \frac{1}{u_j u_1} I_{2j,1}(x_a;0)\, J_{2,2j-1}(x_b;\tau)$$

$$\sim \frac{\rho^2}{4}\Bigg(-\frac{8 \sin 2\pi\rho x_a}{2\pi\rho x_a} \sin \pi\rho x_b e^{(\pi\rho)^2\tau/2}$$

$$\times \int_0^\infty du\, e^{-2(\pi\rho)^2 u^2\tau} \sin 2\pi\rho u x_b \qquad (5.1c)$$

$$-\frac{8 \sin 2\pi\rho x_a}{2\pi\rho x_a} e^{(\pi\rho)^2\tau/2} \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau} \cos \pi\rho x_b(2u+1)\Bigg) \qquad (5.1d)$$

$$\sum_{j,j'=2}^{N/2} \frac{1}{u_j u_{j'}} (I_{2j,2j'}(x_a;0)\, J_{2j'-1,2j-1}(x_b;\tau) + J_{2j,2j'}(x_b;\tau)\, I_{2j'-1,2j-1}(x_a;0))$$

$$\sim -\sum_{j,j'=2}^{N/2} \frac{1}{u_j u_{j'}} I_{2j,2j'-1}(x_a;0)\, J_{2j',2j-1}(x_b;\tau) \tag{5.1e}$$

and

$$\sum_{j'=2}^{N/2} \frac{1}{u_1 u_{j'}} I_{2,2j'}(x_a;0)\, J_{2j'-1,1}(x_b;\tau) + \sum_{j=2}^{N/2} \frac{1}{u_j u_1} I_{2j-1,1}(x_a;0)\, J_{2,2j}(x_b;\tau)$$

$$\sim -\left(\sum_{j'=2}^{N/2} \frac{1}{u_1 u_{j'}} I_{2,2j'-1}(x_a;0)\, J_{2j',1}(x_b;\tau) + \sum_{j=2}^{N/2} \frac{1}{u_j u_1} I_{2j,1}(x_a;0)\, J_{2,2j-1}(x_b;\tau) \right) \tag{5.1f}$$

One of the reasons for the tediousness of this calculation is the need to consider separately the sums in (5.1c) and (5.1e). This is due to the special form of $U_1(x;\tau)$.

Combining these asymptotic results gives as our final formula

$$\begin{aligned}
&{}_d\rho_{(2)}^T((x_a,0),(x_b,\tau)) \\
&= -4\rho^2 \frac{\sin 2\pi\rho x_a \sin \pi\rho x_b}{2\pi\rho x_a} e^{(\pi\rho)^2\tau/2} \int_{1/2}^{\infty} du\, e^{-2(\pi\rho)^2 u^2 \tau} \sin 2\pi\rho u x_b \\
&\quad -8\rho^2 \sin \pi\rho x_a \int_0^{1/2} du\, e^{2(\pi\rho)^2 u^2 \tau} \sin 2\pi\rho u x_b \sin 2\pi\rho u x_a \\
&\quad \times \int_{1/2}^{\infty} dt\, e^{-2(\pi\rho)^2 t^2 \tau} \sin 2\pi\rho t x_b - \frac{8\rho^2 e^{(\pi\rho)^2\tau/2}}{2\pi\rho x_a} \cos \pi\rho x_a \sin \pi\rho x_b \\
&\quad \times \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2 \tau} \sin 2\pi\rho u x_b \sin 2\pi\rho u x_a \\
&\quad +4\rho^2 \int_0^{1/2} dt\, e^{2(\pi\rho)^2 t^2 \tau} \cos 2\pi\rho t(x_a - x_b) \\
&\quad \times \int_{1/2}^{\infty} du\, e^{-2(\pi\rho)^2 u^2 \tau} \cos 2\pi\rho u(x_a - x_b) \\
&\quad +4\rho^2 \int_0^{1/2} dt\, e^{2(\pi\rho)^2 t^2 \tau} \cos 2\pi\rho t(x_a + x_b) \\
&\quad \times \int_{1/2}^{\infty} du\, e^{-2(\pi\rho)^2 u^2 \tau} \cos 2\pi\rho u(x_a + x_b)
\end{aligned} \tag{5.2}$$

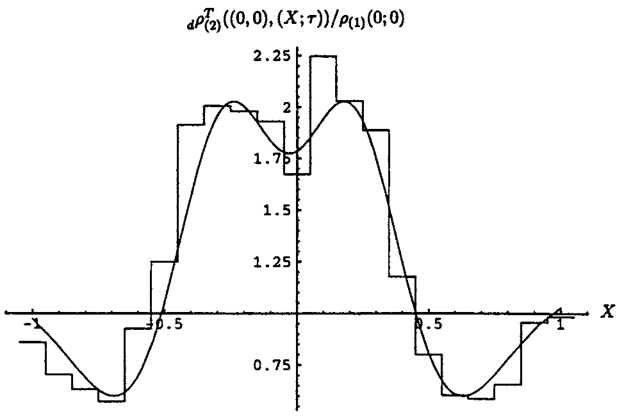

Fig. 3. Same as in Fig. 2a for the quantity ${}_d\rho^T_{(2)}((0, 0), (X; \tau))/\rho_{(1)}(0; 0)$ with $\tau = 0.025$. The bar graph was generated from 3,000 pairs of random matrices of dimension 12×12.

The density–density correlation (5.2) can be calculated empirically by generating pairs of random matrices in an analogous fashion to the calculation of ${}_d\rho^T_{(2)}$ for the GOE → GUE transition (recall the text above (2.24)). The first member of each pair is a Gaussian anti-symmetric Hermitian random matrix generated according to (3.1). However, since the corresponding density is not uniform in the neighborhood of the origin, we must reject the matrix unless there is a scaled eigenvalue in some interval $[Y, Y + dY]$ (recall the discussion of the calculation of (3.27)). We chose $Y = 0$ and $dY = 0.1$. The second member is constructed from the first (assumed accepted) according to Definition 2.1 with a specific value of τ. Analogous to (3.27) we compute the density–density correlation using the formula

$$\frac{{}_d\rho^T_{(2)}((0, 0), (X, \tau))}{\rho_{(1)}(0; \tau)} = -\rho_{(1)}(X; \tau) + \sum_{j = -(N-1)/2}^{(N-1)/2} P(j; X) \tag{5.3}$$

where $P(j; X)$ has the same meaning as in (3.27). The result of this computation for $\tau = 0.025$ and $N = 12$ is given as a bar graph in Fig. 3 and compared with the theoretical curve given by (5.2) with $\rho = 1$ and τ replaced by $2N\tau/\pi^2$.

N **odd**

For N even the sum (5.1a) remains applicable. For the other sums we find

$$\sum_{j,j'=2}^{(N-1)/2} \frac{1}{u_j u_{j'}} I_{2j,2j'-1}(x_a;0)\, J_{2j',2j-1}(x_b;\tau)$$

$$\sim -\frac{\rho^2}{4}\Bigg(-\frac{8}{2\pi\rho x_a} e^{(\pi\rho)^2\tau/2} \sin 2\pi\rho x_a \cos \pi\rho x_b \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u x_b$$

$$+16 \cos \pi\rho x_a \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u x_b$$

$$\times \int_0^{1/2} dt\, e^{2(\pi\rho)^2 t^2\tau} \sin 2\pi\rho t x_b \sin 2\pi\rho t x_a - \frac{16}{2\pi\rho x_a} e^{(\pi\rho)^2\tau/2} \cos \pi\rho x_a \cos \pi\rho x_b$$

$$\times \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u x_b \sin 2\pi\rho u x_a$$

$$-8 \int_0^{1/2} dt\, e^{2(\pi\rho)^2 t^2\tau} \cos 2\pi\rho t(x_a - x_b) \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u(x_a - x_b)$$

$$-8 \int_0^{1/2} dt\, e^{2(\pi\rho)^2 t^2\tau} \cos 2\pi\rho t(x_a + x_b)$$

$$\times \int_0^{1/2} du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u(x_a + x_b)\Bigg) \tag{5.4a}$$

$$2\sum_{j,j'=2}^{(N-1)/2} \frac{1}{u_j u_{j'}} I_{2j,2j'-1}(x_a;0)\, J_{2j',2j-1}(x_b;\tau)$$

$$\sim -\sum_{j,j'=2}^{(N-1)/2} \frac{1}{u_j u_{j'}} \big(I_{2j,2j'}(x_a;0)\, J_{2j'-1,2j-1}(x_b;\tau)$$

$$+ J_{2j,2j'}(x_b;\tau)\, I_{2j'-1,2j-1}(x_a;0)\big) \tag{5.4b}$$

$$\sum_{j'=2}^{(N-1)/2} \frac{1}{u_1 u_{j'}} I_{2,2j'-1}(x_a;0)\, J_{2j',1}(x_b;\tau)$$

$$+\sum_{j=2}^{(N-1)/2} \frac{1}{u_j u_1} I_{2j,1}(x_a;0)\, J_{2,2j-1}(x_b;\tau)$$

$$\sim -\Bigg(\sum_{j'=2}^{(N-1)/2} \frac{1}{u_1 u_{j'}} I_{2,2j'}(x_a;0)\, J_{2j'-1,1}(x_b;\tau)$$

$$+\sum_{j=2}^{(N-1)/2} \frac{1}{u_j u_1} I_{2j-1,1}(x_a;0)\, J_{2,2j}(x_b;\tau)\Bigg) \sim 0 \tag{5.4c}$$

$$\sum_{j=1}^{(N-1)/2} \frac{1}{u_N u_j} I_{2j,N}(x_a;0)\, J_{N+1,2j+1}(x_b;\tau)$$

$$- \sum_{j=1}^{(N-1)/2} \frac{1}{u_j u_N} I_{N,2j-1}(x_a;0)\, J_{2j,N+1}(x_b;\tau)$$

$$\sim -\frac{4\rho^2}{2\pi\rho x_a} \sin 2\pi\rho x_a \cos \pi\rho x_b e^{(\pi\rho)^2\tau/2} \int_0^\infty e^{-2(\pi\rho)^2\tau s^2} \cos 2\pi\rho s x_b \, ds$$

$$+ 8\rho^2 \cos \pi\rho x_a \int_0^\infty e^{-2(\pi\rho)^2 s^2\tau} \cos \pi\rho x_a$$

$$\times \int_0^{1/2} e^{2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u x_a \cos 2\pi\rho u x_b \, du \tag{5.4d}$$

Combining these asymptotic results gives that in the thermodynamic limit, for N odd

$${}_d\rho_{(2)}^T((x_a,0),(x_b,\tau))$$

$$= 4\rho^2 \frac{\sin 2\pi\rho x_a \cos \pi\rho x_b}{2\pi\rho x_a} e^{(\pi\rho)^2\tau/2} \int_{1/2}^\infty du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u x_b$$

$$- 8\rho^2 \cos \pi\rho x_a \int_0^{1/2} du\, e^{2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u x_b \cos 2\pi\rho u x_a$$

$$\times \int_{1/2}^\infty dt\, e^{-2(\pi\rho)^2 t^2\tau} \cos 2\pi\rho t x_b - \frac{8\rho^2 e^{(\pi\rho)^2\tau/2}}{2\pi\rho x_a} \cos \pi\rho x_b \sin \pi\rho x_a$$

$$\times \int_{1/2}^\infty du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u x_b \cos 2\pi\rho u x_a$$

$$+ 4\rho^2 \int_0^{1/2} dt\, e^{2(\pi\rho)^2 t^2\tau} \cos 2\pi\rho t(x_a - x_b)$$

$$\times \int_{1/2}^\infty du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u(x_a - x_b)$$

$$+ 4\rho^2 \int_0^{1/2} dt\, e^{2(\pi\rho)^2 t^2\tau} \cos 2\pi\rho t(x_a + x_b)$$

$$\times \int_{1/2}^\infty du\, e^{-2(\pi\rho)^2 u^2\tau} \cos 2\pi\rho u(x_a + x_b)\Big) \tag{5.5}$$

As a check on this formula, we note that for N odd, $\rho_{(1)}(0;0)=0$, so (5.3) gives that ${}_d\rho^T_{(2)}((0,0),(x,\tau))=0$. It is a simple exercise to verify that (5.5) has this property.

Further checks on both (5.2) and (5.5) are possible. In the limit $\tau \to 0$ with $x_a \neq x_b$ we must have

$$ {}_d\rho^T_{(2)}((x_a,0),(x_b,\tau)) \sim \rho^{T(\mathrm{AH})}_{(2)}(x_a,x_b) \tag{5.6} $$

where $\rho^{T(\mathrm{AH})}_{(2)}$ denotes the scaled truncated two-point distribution function for Gaussian antisymmetric Hermitian random matrices (for N odd this statement also assumes $x_a, x_b \neq \tau$). Explicitly[4]

$$ \rho^{T(\mathrm{AH})}_{(2)}(x_a,x_b) = -\rho^2\left(\frac{\sin \pi\rho(x_a-x_b)}{\pi\rho(x_a-x_b)} + (-1)^N \frac{\sin \pi\rho(x_a+x_b)}{\pi\rho(x_a+x_b)}\right)^2 \tag{5.7} $$

To verify the property (5.6) we use the formulas

$$ \lim_{\varepsilon\to 0}\int_{1/2}^{\infty} e^{-\varepsilon s^2}\cos 2\pi s\rho x\, ds = \frac{\sin \pi\rho x}{2\pi\rho x}, \qquad x\neq 0 $$

and

$$ \lim_{\varepsilon\to 0}\int_{1/2}^{\infty} e^{-\varepsilon s^2}\sin 2\pi s\rho x\, ds = \frac{\cos \pi\rho x}{2\pi\rho x}, \qquad x\neq 0 $$

in (5.2) and (5.5). After some manipulation the expression (5.7) is obtained.

6. CONCLUDING REMARKS

The density–density correlation and n-point equal parameter distribution can be computed exactly for parameter-dependent Gaussian random matrices intermediate between symmetric and Hermitian, and anti-symmetric Hermitian and Hermitian. The same correlations can also be computed for the circular ensemble analogues of the these matrices.

Although the equal-parameter distributions, for finite N in the Gaussian case at least, have been calculated previously, we have demonstrated how all correlations can be the calculated through the use of general formulas based on skew-orthogonal polynomials. Furthermore, since parameter-dependent random matrices are very simple to generate on a computer, we have been able to provide the empirical evaluation of some these correlations and thereby to realize the corresponding theoretical predictions, up to statistical accuracy.

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